ABELIAN EXTENSIONS OF ALGEBRAS IN CONGRUENCE-MODULAR VARIETIES

WILLIAM H. ROWAN

Abstract. We define abelian extensions of algebras in congruence-modular varieties. The theory is sufficiently general that it includes, in a natural way, extensions of R-modules for a ring R. We also define a cohomology theory, which we call clone cohomology, such that the cohomology group in dimension one is the group of equivalence classes of extensions.

Introduction

The theory of abelian extensions of algebras has a long history, going from abelian extensions of groups and modules to abelian extensions of algebras in arbitrary varieties. Extensions of modules and abelian extensions of groups and Lie algebras are standard topics in treatises on Homological Algebra. The similarity of the situation for groups and Lie algebras points to a common generalization, which is our goal in this paper. Previous attempts at such a generalization, and related work, can be found in [6], [9], [1], [3], [2], [12], [13], [5], [16].

The most general treatment of which we are aware, in [3], treats abelian extensions in the generality of an algebra in an arbitrary variety of algebras, and a Beck module over that algebra. (A *Beck module* over A in a variety $\mathbf{V}$ is an abelian group object in the category $(\mathbf{V} \downarrow A)$ of algebras over A.)

Our treatment is less general because we require $\mathbf{V}$ to be congruence-modular. However, the apparatus of commutator theory, in particular, the existence of a difference term, allows us to prove an important lemma (lemma 3.2) which lets us work with a simpler and more conceptual definition of an extension than that in [3]. We are also able to treat module extensions with the same theory, in a fairly natural way.

In both cases, there is an associated cohomology theory, the cohomology group in dimension one being the group of equivalence classes of extensions. We have shown that these groups are isomorphic for $\mathbf{V}$ congruence-modular, but will not give the proof in this paper as it is quite tedious. The theory in [3] (and also [1] and [2]) is called *comonadic cohomology*, and we call ours *clone cohomology*.

We frame our theory not in terms of algebras over A, and Beck modules over A, but of objects in equivalent categories we call the category of A-overalgebras and the category of

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abelian group A-overalgebras. We will briefly explain our reasons for doing so at the end of §1, where these categories are defined.

The theory of abelian extensions is interesting because it organizes, from a certain point of view, the possible structures of a class of algebras, related to A (or an A-overalgebra Q) and an abelian group A-overalgebra M, into an abelian group, functorial in a way that we will discuss in §9. Also, that abelian group is a cohomology group for a suitable cohomology theory derived from A and M.

After a section of preliminaries, §1 of the paper defines the categories of A-overalgebras and abelian group A-overalgebras. §2 sketches the theory of enveloping ringoids very briefly, and is included to show how that theory can be applied to constructing abelian group A-overalgebras free on an A-tuple of sets of generators. §3 defines abelian extensions and performs some preliminary analyses of them. §4 defines and explores the formalism of factor sets of extensions. §5 introduces the definition of equivalence of extensions, and defines the set $\mathbf{E_V}(A, M)$ of equivalence classes of extensions. §6 then shows how the set of equivalence classes of extensions can be seen as a cohomology group. §7 explores composition operations between abelian extensions and homomorphisms, and §8 discusses the group law in the set of equivalence classes of extensions. §9 contains a reformulation of the definition that defines $\mathbf{E_V}(Q, M)$ for an A-overalgebra Q and abelian group A-overalgebra M, functorially in Q and M. §10 shows how module extensions can be treated. §11 presents a cohomology theory we call *clone cohomology* because its definition intimately involves the clone of the variety $\mathbf{V}$ to which A, Q, and M all belong. §12 explores varying the variety $\mathbf{V}$ used in defining clone cohomology, giving *relative clone cohomology.* Finally, we pose a number of questions that seem important to ask about this theory, and about the relationship of clone cohomology and comonadic cohomology.

0. Preliminaries

Category theory. We follow [10] in terminology and notation.

Homological algebra. We assume a familiarity with concepts and conventions of homological algebra, such as can be found in [11], [8], and [17].

Universal algebra. We assume the basic definitions of universal algebra, such as can be found in [4], are known to the reader. Unlike some authors, we admit the possibility that an algebra can have an empty underlying set.

The kernel congruence of a homomorphism f will be denoted by $\ker f$. The other sort of kernel, $f^{-1}(0)$, will be denoted by $\operatorname{Ke} f$.

We denote the greatest and least congruences of A by $\top_A$ and $\bot_A$, and the identity homomorphism of A by 1_A. Basic operations or term operations of A will occasionally be denoted by ω^A or t^A, but we almost always drop the superscript.

If A stands for an algebra, we use $U(A)$ to stand for the underlying set of the algebra.

Clones. A *clone* is an $\mathbb{N}$-tuple of sets V_n (the n-ary elements of the clone V) such that for each n, and each i with $1 \le i \le n$, there is an element $\pi^V_{in} \in V_n$, called the i^{th} *of* n *projection*, and such that, for each n, each n'-tuple $\mathbf{v}$ of elements of V_n, and each $v' \in V_{n'}$, there is an element $v'\mathbf{v} \in V_n$, called the *clone composite* of v' and $\mathbf{v}$, satisfying

1. $\pi^V_{in}\mathbf{v} = v_i$,
2. $v\langle \pi^V_{1n}, \dots, \pi^V_{nn}\rangle = v$, and
3. $u(\mathbf{vw}) = (u\mathbf{v})\mathbf{w}$,

whenever the relevant compositions are defined. ($\mathbf{vw}$ stands for $\langle v_1\mathbf{w}, \dots, v_n\mathbf{w}\rangle$ if $\mathbf{v}$ is an n-tuple.)

As an example of a clone, given a set S, we have the *clone of* S, denoted by $\operatorname{Clo} S$. $\operatorname{Clo}_n S$ is the set of n-ary functions from S to S, $\pi^{\operatorname{Clo} S}_{in}$ is the n-ary function on S choosing the i^{th} of its n arguments, and given n, n', an n'-tuple of n-ary functions $\mathbf{f}$, and an n'-ary function f', the clone composite $f'\mathbf{f}$ is the function defined by

$$\mathbf{s} \mapsto f'(f_1(\mathbf{s}), \dots, f_{n'}(\mathbf{s})).$$

Another example of a clone is the clone of a variety $\mathbf{V}$, denoted by $\operatorname{Clo} \mathbf{V}$. Elements of $\operatorname{Clo}_n \mathbf{V}$ are equivalence classes of n-ary term operations of the algebras in $\mathbf{V}$, where terms t and t' are equivalent if $t(\mathbf{x}) = t'(\mathbf{x})$ is an identity of $\mathbf{V}$.

If V, V' are clones, a *homomorphism of clones* from V to V' is an $\mathbb{N}$-tuple f of functions $f_n : V_n \to V'_n$, such that for all i and n, $f_n(\pi^V_{in}) = \pi^{V'}_{in}$, and for all n and n', $\mathbf{v} \in V_n^{n'}$, and $v' \in V_{n'}$, we have $f_{n'}(v')f_n(\mathbf{v}) = f_n(v'\mathbf{v})$.

An algebra A in a variety $\mathbf{V}$ is the same as a clone homomorphism from $\operatorname{Clo} V$ to $\operatorname{Clo} S$, where S is the underlying set of A.

A clone V can be viewed as a category with one object for each natural number. The arrows from n to n' are n'-tuples of elements of V_n, and the identity of n is $\langle \pi^V_{1n}, \dots, \pi^V_{nn}\rangle$. In the resulting category, n is the n-fold direct power of 1. Thus, the category is what is often called a *theory*.

The modular commutator. In a congruence-modular variety $\mathbf{V}$ (i.e., such that for all $A \in \mathbf{V}$, $\operatorname{Con} A$ is a modular lattice) the congruence lattices admit a binary operation, called the *commutator*, that generalizes some well-known operations such as the commutator of two normal subgroups of a group. A comprehensive treatment of the theory of this operation, and related matters, can be found in [7].

The commutator of two congruences α, $\beta \in \operatorname{Con} A$ is denoted by $[\alpha, \beta]$. A congruence α is said to be *abelian* if $[\alpha, \alpha] = \bot_A$.

One definition of the commutator is as follows: If $A \in \mathbf{V}$, a congruence-modular variety of algebras, and θ, $\psi \in \operatorname{Con} A$, then $[\theta, \psi]$ is the least congruence such that for all n, for all $(n+1)$-ary terms t, for all a, $a' \in A$ such that $a \mathrel{\theta} a'$, and for all $\mathbf{b}$, $\mathbf{c} \in A^n$ such that $b_i \mathrel{\psi} c_i$ for all i, we have

$$t(a, \mathbf{b}) \mathrel{[\theta, \psi]} t(a, \mathbf{c})$$

implies

$$t(a', \mathbf{b})\ [\theta, \psi]\ t(a', \mathbf{c}).$$

Difference terms. If $\mathbf{V}$ is a congruence-modular variety of algebras, a ternary term d is called a *difference term* for $\mathbf{V}$ if

1. $d(x, x, y) = y$ is an identity of $\mathbf{V}$, and
2. for all $A \in \mathbf{V}$, $\theta \in \operatorname{Con} A$, and x, $y \in A$ such that $x\ \theta\ y$, we have $d(x, y, y)\ [\theta, \theta]\ x$.

At least one such term exists for any congruence-modular variety.

V-objects in a category. If $\mathbf{C}$ is a category, and $\mathbf{V}$ is a variety of algebras, a $\mathbf{V}$-object in $\mathbf{C}$ is a pair $\langle c, F\rangle$, consisting of an object $c \in \mathbf{C}$, and a contravariant functor $F : \mathbf{C} \to \mathbf{V}$, such that $UF = \mathbf{C}(-, c)$, where $U : \mathbf{V} \to \mathbf{Set}$ is the forgetful functor. If $\langle c, F\rangle$ and $\langle c', F'\rangle$ are $\mathbf{V}$-objects, a *homomorphism of* $\mathbf{V}$*-objects from* $\langle c, F\rangle$ *to* $\langle c', F'\rangle$ is an arrow $f : c \to c'$ such that for each object $d \in \mathbf{C}$, the function $\mathbf{C}(d, f) : \mathbf{C}(d, c) \to \mathbf{C}(d, c')$ is the underlying function of a (necessarily unique, since U is faithful) arrow $\bar{f} : F(d) \to F'(d)$.

$\mathbf{V}$-objects in $\mathbf{C}$, and the homomorphisms between them, form a category in an obvious manner, which we denote by $\mathbf{V}[\mathbf{C}]$.

1. The Categories A-$\mathbf{Set}$, $\mathbf{Ov}[A, \mathbf{V}]$, and $\mathbf{Ab}[A, \mathbf{V}]$

The category of A-sets. Let A be an algebra. We define an *A-set* to be a $U(A)$-tuple of sets. If S is an A-set, we will denote the member of the tuple corresponding to an element $a \in A$ by ${}_aS$. If S, S' are A-sets, an *A-function* from S to S' is a $U(A)$-tuple f such that the element corresponding to each $a \in A$, which we denote by ${}_af$, is a function from ${}_aS$ to ${}_aS'$. We write $f : S \to S'$. A-sets and A-functions form a category, A-$\mathbf{Set}$, in an obvious manner.

Overalgebras. If A is an algebra, an *A-overalgebra* is an A-set Q, such that for each n, and each n-ary basic operation ω of the type of A, there is a $U(A)^n$-tuple of functions $\omega^Q_{\mathbf{a}} : {}_{a_1}Q \times \ldots \times {}_{a_n}Q \to {}_{\omega(\mathbf{a})}Q$.

In what follows, we will write ${}_{\mathbf{a}}Q$ for the product ${}_{a_1}Q \times \ldots \times {}_{a_n}Q$. Thus, $\omega^Q_{\mathbf{a}} : {}_{\mathbf{a}}Q \to {}_{\omega(\mathbf{a})}Q$.

If Q, Q' are A-overalgebras, a *homomorphism of A-overalgebras from Q to Q'* is an A-function $f : Q \to Q'$ such that for each n, each n-ary basic operation ω, each $\mathbf{a} \in A^n$, and each $\mathbf{q} \in {}_{\mathbf{a}}Q$, we have

$${}_{\omega(\mathbf{a})}f(\omega^Q_{\mathbf{a}}(\mathbf{q})) = \omega^{Q'}_{\mathbf{a}}({}_{\mathbf{a}}f(\mathbf{q})),$$

where ${}_{\mathbf{a}}f(\mathbf{q})$ stands for $\langle {}_{a_1}f(q_1), \ldots, {}_{a_n}f(q_n)\rangle$.

A-overalgebras and their homomorphisms form a category in an obvious manner, which we denote by $\mathbf{Ov}[A]$.

For an example of an A-overalgebra, let $\langle B, \pi\rangle$ be an object of the "comma category" $(\Omega\text{-}\mathbf{Alg} \downarrow A)$ of algebras over A. That is, let B be an algebra of the same type Ω as A, and let $\pi : B \to A$ be a homomorphism. Then we define the A-overalgebra $[\![B, \pi]\!]$ by ${}_a[\![B, \pi]\!] = \pi^{-1}(a)$ and $\omega^{[\![B,\pi]\!]}_{\mathbf{a}}(\mathbf{b}) = \omega^B(\mathbf{b})$.

If Q is an A-overalgebra, we define the *total algebra* of Q, denoted by $A \ltimes Q$, to be the set of pairs $\{ \langle a, q\rangle : a \in A, q \in {}_aQ \}$, provided with operations defined by

$$\omega(\langle a_1, q_1\rangle, \dots , \langle a_n, q_n\rangle) = \langle \omega(\mathbf{a}), \omega^Q_{\mathbf{a}}(\mathbf{q})\rangle;$$

we say that an A-overalgebra Q is *totally in* $\mathbf{V}$, where V is a variety of algebras of the type of A, if $A \ltimes Q \in \mathbf{V}$.

If Q is an A-overalgebra, then accompanying the total algebra $A \ltimes Q$ there is a homomorphism $\pi_Q : A \ltimes Q \to A$, defined by $\langle a, q\rangle \mapsto a$. It is clear that $\mathbf{Ov}[A, \mathbf{V}]$ is equivalent as a category to the category $(\mathbf{V} \downarrow A)$. One leg of an equivalence takes an A-overalgebra Q to $\langle A \ltimes Q, \pi_Q\rangle$. The other leg takes an algebra over A, $\langle B, \pi\rangle$, to $[\![B, \pi]\!]$.

There is an evident forgetful functor from $\mathbf{Ov}[A, \mathbf{V}]$ to A-**Set**. To construct an A-overalgebra free on an A-set S (i.e., the value of the corresponding left adjoint functor) form a free algebra F on a disjoint union of the ${}_aS$, (a free algebra in $\mathbf{V}$, that is, also known as a *relatively free* algebra), and map the generators to A in the obvious way, giving a homomorphism $\pi : F \to A$. The A-overalgebra $[\![F, \pi]\!]$ is then free on S.

One-one and onto homomorphisms of A-overalgebras. If f is a homomorphism of A-overalgebras, then we say that f is *one-one* if each ${}_af$ is one-one, and we say that f is *onto* if each ${}_af$ is onto. If f is both one-one and onto, then it is an isomorphism in the category of A-overalgebras.

A-operations. Let A be a set, and ω an n-ary operation on A. If S is an A-set, then an *A-operation on S, over* ω, is a $U(A)^n$-tuple ω' of functions $\omega'_{\mathbf{a}} : {}_{\mathbf{a}}S \to {}_{\omega(\mathbf{a})}S$. We can now rephrase our definition of an A-overalgebra: it is an A-set Q, together with an A-operation ω^Q over ω^A for every basic operation ω on A.

If Q is an A-overalgebra totally in $\mathbf{V}$, and $v \in \operatorname{Clo}_n \mathbf{V}$, then we can define an A-operation v^Q over v^A by letting $v^Q_{\mathbf{a}}(\mathbf{q})$ be the second component of the pair

$$v^{A\ltimes Q}(\langle a_1, q_1\rangle, \dots , \langle a_n, q_n\rangle);$$

we will thus use v^Q to denote that A-operation, regardless of whether v is an n-ary basic operation, n-ary term operation, or n-ary element of $\operatorname{Clo} \mathbf{V}$.

If A is a set, and S an A-set, then we can define $\operatorname{Clo}^A S$, the *clone of A-operations on S* to be, for each n, the set of pairs $\langle \omega, \omega'\rangle$ where ω is an n-ary operation on A, and ω' is an n-ary A-operation on S over ω. Projections and clone composition are easy to define, and there is a clone homomorphism $\pi : \operatorname{Clo}^A S \to \operatorname{Clo} A$ given by taking the first component of each pair.

If A is an algebra in a variety $\mathbf{V}$, ϕ^A is the corresponding clone homomorphism from $\operatorname{Clo} \mathbf{V}$ to $\operatorname{Clo} A$, and Q is an A-set, then an A-overalgebra structure on Q, totally in $\mathbf{V}$, just amounts to a clone homomorphism $\phi^Q : \operatorname{Clo} \mathbf{V} \to \operatorname{Clo}^A Q$, such that $\pi\phi^Q = \phi^A$. Accordingly, to define an A-overalgebra, it suffices to define v^A for all v, with the condition that the given definitions provide a well-defined clone homomorphism.

Pointed overalgebras. If A is an algebra, a *pointed A-overalgebra* is an A-overalgebra P, such that each ${}_aP$ has a basepoint ${}_a*^P$, or simply ${}_a*$, such that for each basic operation ω, n-ary, and each $\mathbf{a} \in A^n$, $\omega^P_{\mathbf{a}}({}_{a_1}*, \dots, {}_{a_n}*) = {}_{\omega(\mathbf{a})}*$.

If P, P' are pointed A-overalgebras, a *homomorphism from P to P'* is a homomorphism of A-overalgebras $f : P \to P'$ such that for each a, ${}_af({}_a*^P) = {}_a*^{P'}$.

Pointed overalgebras totally in $\mathbf{V}$, and the homomorphisms between them, form a category $\mathbf{Pnt}[A, \mathbf{V}]$ in an obvious manner. As the notation suggests, it is precisely the category $\mathbf{Pnt}[\mathbf{Ov}[A, \mathbf{V}]]$ of pointed set objects in the category $\mathbf{Ov}[A, \mathbf{V}]$.

As an example of a pointed A-overalgebra, let $\alpha \in \operatorname{Con} A$. We define a pointed A-overalgebra α^* by ${}_a\alpha^* = \{\, a' : a \;\alpha\; a' \,\}$, ${}_a* = a$, and $\omega^{\alpha^*}_{\mathbf{a}}(\mathbf{c}) = \omega(\mathbf{c})$ for $\mathbf{c} \in {}_{\mathbf{a}}\alpha^*$.

If P is a pointed A-overalgebra, then accompanying $A \ltimes P$ and π_P there is a homomorphism $\iota_P : A \to A \ltimes P$, defined by $\iota_P : a \mapsto \langle a, {}_a* \rangle$. The triple $\langle A \ltimes P, \pi_P, \iota_P \rangle$ can be viewed as a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\iota_P"] \arrow[d, equal] & A \ltimes P \arrow[d, "\pi_P;"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

in slightly different terms, it is a pointed set object in the category of algebras over A.

Given such a diagram, or, a triple $\langle B, \pi, \iota \rangle$ with $\pi : B \to A$, $\iota : A \to B$, and $\pi\iota = 1_A$, we denote by $[\![B, \pi, \iota]\!]$ the pointed A-overalgebra with underlying A-overalgebra $[\![B, \pi]\!]$ and basepoints $\iota(a) \in {}_a[\![B, \pi]\!]$.

The constructions $P \mapsto \langle A \ltimes P, \pi_P, \iota_P \rangle$ and $\langle B, \pi, \iota \rangle \mapsto [\![B, \pi, \iota]\!]$ are clearly two legs of an equivalence between the categories $\mathbf{Pnt}[A, \mathbf{V}]$ and $\mathbf{Pnt}(\mathbf{V} \downarrow A)$.

There is an obvious forgetful functor from $\mathbf{Pnt}[A, \mathbf{V}]$ to $\mathbf{Ov}[A, \mathbf{V}]$. A corresponding free functor can be defined as follows: given an A-overalgebra Q, form the algebra $B = A \coprod (A \ltimes Q)$. Define $\pi : B \to A$ by applying the universal property of the coproduct to the homomorphisms 1_A and π_Q. Define $\iota : A \to B$ as the insertion of A into the coproduct. Then $[\![B, \pi, \iota]\!]$ is a pointed A-overalgebra free on Q.

Abelian group overalgebras. An *abelian group A-overalgebra* is an A-overalgebra M such that on each ${}_aM$ there is the structure of an abelian group, in such a way that the functions $\omega^M_{\mathbf{a}} : {}_{\mathbf{a}}M \to {}_{\omega(\mathbf{a})}M$ are abelian group homomorphisms. If M, M' are abelian group A-overalgebras, a *homomorphism of abelian group A-overalgebras from M to M'* is an A-overalgebra homomorphism $f : M \to M'$ such that each ${}_af$ is an abelian group homomorphism.

Abelian group A-overalgebras totally in $\mathbf{V}$, and the homomorphisms between them, form a category in an obvious manner, which we denote by $\mathbf{Ab}[A, \mathbf{V}]$. It is the category of abelian group objects in $\mathbf{Ov}[A, \mathbf{V}]$.

Categorical algebraists have given the term *Beck module over A* to an abelian group object in the category $(\mathbf{V} \downarrow A)$. Clearly, $\mathbf{Ab}[A, \mathbf{V}]$ is equivalent to the category of Beck modules over A.

There is an obvious forgetful functor from $\mathbf{Ab}[A, \mathbf{V}]$ to $\mathbf{Pnt}[A, \mathbf{V}]$.

Theorem 1.1. ([14]) *Let* $\mathbf{V}$ *be a congruence-modular variety of algebras, and* $A \in \mathbf{V}$*. Let* P *be a pointed* A*-overalgebra which is the underlying pointed* A*-overalgebra of an abelian group overalgebra. Then there is a unique assignment of abelian group structures to the pointed sets* ${}_aP$*, such that* ${}_a*$ *is the zero element of each* ${}_aP$ *and the functions* $\omega^P_{\mathbf{a}}$ *are abelian group homomorphisms. The group operations in* ${}_aP$ *satisfy* $p - p' + p'' = d^P_{\langle a,a,a\rangle}(p, p', p'')$*.*

Abelian group A-overalgebras free on a pointed A-overalgebra. If $\mathbf{V}$ is congruence-modular, a free functor (left adjoint) for the forgetful functor from $\mathbf{Ab}[A, \mathbf{V}]$ to $\mathbf{Pnt}[A, \mathbf{V}]$ is as follows: Given a pointed A-overalgebra P, draw the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\iota_P} & A \ltimes P & \xrightarrow{\mathrm{nat}[\kappa,\kappa]} & (A \ltimes P)/[\kappa, \kappa] \\ \| & & \downarrow{\scriptstyle \pi_P} & & \downarrow{\scriptstyle \pi,} \\ A & = & A & = & A \end{array}$$

where $\kappa = \ker \pi_P$ and π is the unique homomorphism making the diagram commute.

$$M = [\![(A \ltimes P)/[\kappa, \kappa], \pi, \mathrm{nat}[\kappa, \kappa] \circ \iota_P]\!]$$

is then a pointed overalgebra, which is an abelian group overalgebra in a unique way, by theorem 1.1. M is an abelian group A-overalgebra free on P.

$\mathbf{V}'$ A-overalgebras. Suppose $\mathbf{V}'$ is another variety of algebras than $\mathbf{V}$, perhaps of a different type. We define a $\mathbf{V}'$ A-overalgebra to be an A-overalgebra M such that the ${}_aM$ are algebras in $\mathbf{V}'$ and the $\omega^M_{\mathbf{a}}$ are homomorphisms of $\mathbf{V}'$, and a homomorphism of $\mathbf{V}'$ A-overalgebras to be an A-overalgebra homomorphism f such that the ${}_af$ are homomorphisms of algebras in $\mathbf{V}'$. We denote the category of $\mathbf{V}'$ A-overalgebras totally in $\mathbf{V}$, and homomorphisms betwen them, by $\mathbf{V}'[A, \mathbf{V}]$.

The category $\mathbf{V}'[A, \mathbf{V}]$ generalizes $\mathbf{Pnt}[A, \mathbf{V}]$ and $\mathbf{Ab}[A, \mathbf{V}]$ in an obvious way, and we have $\mathbf{V}'[A, \mathbf{V}] = \mathbf{V}'[\mathbf{Ov}[A, \mathbf{V}]]$, the category of $\mathbf{V}'$-objects in the category $\mathbf{Ov}[A, \mathbf{V}]$.

Although $\mathbf{V}'[A, \mathbf{V}]$ is the category of $\mathbf{V}'$-objects in the category $\mathbf{Ov}[A, \mathbf{V}]$, it is also true that given $M \in \mathbf{V}'[A, \mathbf{V}]$, and an A-set S, the A-functions from S to M form an algebra of $\mathbf{V}'$ in a natural way.

If $M \in \mathbf{V}'[A, \mathbf{V}]$, and u is an n-ary basic operation, term operation, or element of $\mathrm{Clo}\,\mathbf{V}'$, we will denote by ${}_au$, or occasionally by ${}_au^M$, that operation on the algebra ${}_aM$.

Restriction and induction functors. For each category $\mathbf{V}'[A, \mathbf{V}]$, algebra $X \in \mathbf{V}$, and homomorphism $f : X \to A$, there is a functor ${}_f\mathrm{Res} : \mathbf{V}'[A, \mathbf{V}] \to \mathbf{V}'[X, \mathbf{V}]$, defined by ${}_x({}_f\mathrm{Res}\, M) = {}_{f(x)}M$ and $\omega^{{}_f\mathrm{Res}\, M}_{\mathbf{x}} = \omega^M_{f(\mathbf{x})}$, where $f(\mathbf{x})$ stands for $\langle f(x_1), \dots, f(x_n)\rangle$.

The restriction functors all have left adjoints, which are constructed in [14]. We call these *induction* functors.

We will have occasion to use the functor of induction of abelian group overalgebras in §10.

Products. Let A be an algebra. If $S_1, \ldots, S_n$ are A-sets, then a product $\Pi_i S_i$ in the category of A-sets is given by ${}_a(\Pi_i S_i) = {}_aS_1 \times \ldots \times {}_aS_n$. Similarly, if $M_1, \ldots, M_n \in \mathbf{V}'[A, \mathbf{V}]$, then a product $\Pi_i M_i \in \mathbf{V}'[A, \mathbf{V}]$ is given by ${}_a(\Pi_i M_i) = {}_aM_1 \times \ldots \times {}_aM_n$, by

$$v_{\mathbf{a}}^{\Pi_i M_i}(\mathbf{m}_1, \ldots, \mathbf{m}_k) = \langle v_{\mathbf{a}}^{M_1}(m_{11}, \ldots, m_{k1}), \ldots, v_{\mathbf{a}}^{M_n}(m_{1n}, \ldots, m_{kn})\rangle,$$

for $v \in \operatorname{Clo}_k \mathbf{V}$, and by

$${}_au(\mathbf{m}_1, \ldots, \mathbf{m}_k) = \langle {}_au^{M_1}(m_{11}, \ldots, m_{k1}), \ldots, {}_au^{M_n}(m_{11}, \ldots, m_{k1})\rangle,$$

for $u \in \operatorname{Clo}_k \mathbf{V}'$.

Suppose, on the other hand, that we are given algebras $A_i \in \mathbf{V}$, and objects $M_i \in \mathbf{V}'[A_i, \mathbf{V}]$, for $i = 1, \ldots, n$. Let $A = \Pi_i A_i$. Define the *outer product* $\boxtimes_i M_i \in \mathbf{V}'[A, \mathbf{V}]$ by ${}_{\mathbf{a}}(\boxtimes_i M_i) = \Pi_i({}_{a_i}M_i)$, and by

$$\begin{aligned} &v_{\langle \mathbf{a}_1, \ldots, \mathbf{a}_k\rangle}^{\boxtimes_i M_i}(\mathbf{m}_1, \ldots, \mathbf{m}_k) \\ &\qquad = \langle v_{\langle a_{11}, \ldots, a_{k1}\rangle}^{M_1}(m_{11}, \ldots, m_{k1}), \ldots, v_{\langle a_{1n}, \ldots, a_{kn}\rangle}^{M_n}(m_{1n}, \ldots, m_{kn})\rangle, \end{aligned}$$

for $v \in \operatorname{Clo}_k \mathbf{V}$, and

$${}_{\mathbf{a}}u(\mathbf{m}_1, \ldots, \mathbf{m}_i) = \langle {}_{a_1}u^{M_1}(m_{11}, \ldots, m_{k1}), \ldots, {}_{a_n}u^{M_n}(m_{1n}, \ldots, m_{kn})\rangle,$$

for $u \in \operatorname{Clo}_k \mathbf{V}'$. That is, $M = \Pi_i({}_{\pi_{A,i}}\operatorname{Res} M_i)$, where the $\pi_{A,i} : A \to A_i$ are the projections to the factors.

Theorem 1.2. *If all the A_i are the same algebra A, we have ${}_{\Delta_A}\operatorname{Res}(\boxtimes_i M_i) = M^n$, where the homomorphism $\Delta_A : A \to \Pi_i A$ is defined by $a \mapsto \langle a, \ldots, a\rangle$.*

Advantages of the overalgebra formalism. There are two main advantages for using the formalism of A-overalgebras rather than that of algebras over A. One is that given an object $M \in \mathbf{V}'[A, \mathbf{V}]$, the ${}_aM$, which are algebras of $\mathbf{V}'$, are important objects of study, and a formalism that provides for this is useful. If we use the formalism of algebras over A, then we will end up considering the ${}_aM$ anyway, in a different form, as $\pi^{-1}(a)$ for $a \in A$.

The other main advantage is that the formalism of A-overalgebras allows the ${}_aM$ to be nondisjoint. One way this helps is to facilitate defining and using the pointed A-overalgebra α^* for $\alpha \in \operatorname{Con} A$. If we don't allow nondisjoint sets, then we must work with the algebra often denoted by $A(\alpha) = A \ltimes \alpha^*$, along with the projection π_{α^*} to A.

The other way allowing the ${}_aM$ to be nondisjoint is helpful, and this is probably the most important advantage, is in defining and using the restriction functors. Using the formalism of algebras over A requires that the restrictions be defined as pullbacks, and then frequent use must be made of the universal property of the pullback. This can be done, of course, but it is tedious, and tends to obscure the situation.

2. Enveloping Ringoids

Ringoids. A *ringoid* is a small additive category. If $\mathbf{X}$ is a ringoid, with set of objects A, we write ${}_{a'}\mathbf{X}_a$ for $\mathbf{X}(a, a')$. A left $\mathbf{X}$-*module* is an additive functor from $\mathbf{X}$ to $\mathbf{Ab}$, the category

of abelian groups. If M is a left $\mathbf{X}$-module, we write ${}_aM$ instead of $M(a)$, and if $m \in {}_aM$ and $r \in {}_{a'}\mathbf{X}_a$, we write rm rather than $M(r)(m)$.

If A is an algebra in a variety $\mathbf{V}$, the *enveloping ringoid for A, with respect to* $\mathbf{V}$, is a certain ringoid denoted by $\mathbb{Z}[A, \mathbf{V}]$, which has the underlying set of A as its set of objects, and such that the category of left $\mathbb{Z}[A, \mathbf{V}]$-modules is isomorphic to the category $\mathbf{Ab}[A, \mathbf{V}]$. Enveloping ringoids are treated in detail in [14] and [15].

Construction of the enveloping ringoid. Given an algebra A in a variety $\mathbf{V}$, we construct an object $M_a \in \mathbf{Ab}[A, \mathbf{V}]$ free on an A-set consisting of a singleton at a and the null set elsewhere, for each $a \in A$. (An abelian group A-overalgebra free on an A-set S can be constructed by constructing an A-overalgebra free on S, a pointed A-overalgebra free on that, and an abelian group A-overalgebra free on that.)

The enveloping ringoid can then be given as ${}_{a'}\mathbb{Z}[A, \mathbf{V}]_a = {}_{a'}(M_a)$ for all a, $a' \in A$.

Abelian group A-overalgebras free on an A-set. Once the enveloping ringoid is defined, there is another method available for defining abelian group A-overlaying algebras free on A-sets. Let A be an algebra in the variety $\mathbf{V}$, and let S be an A-set. For each $a \in A$, let ${}_aM$ be the set of finite formal linear combinations

$$\sum_i r_i s_i,$$

where $r_i \in {}_a\mathbb{Z}[A, \mathbf{V}]_{b_i}$ and $s_i \in {}_{b_i}S$. This defines an object $M \in \mathbf{Ab}[A, \mathbf{V}]$ free on S.

3. Abelian Extensions

Let A be an algebra of $\mathbf{V}$, and let $M \in \mathbf{Ab}[A, \mathbf{V}]$. We define an extension in $\mathbf{V}$ of A by M to be a triple $\langle \chi, E, \pi \rangle$ such that E is an algebra of $\mathbf{V}$, $\pi : E \to A$ is an onto homomorphism, and $\chi : {}_\pi\mathrm{Res}\, P \to \kappa^*$ is an isomorphism of pointed overalgebras, where P is the underlying pointed A-overalgebra of M and $\kappa = \ker \pi$.

For example, given $A \in \mathbf{V}$, and M, totally in $\mathbf{V}$, we can form the total algebra $A \ltimes M$. Let $\kappa = \ker \pi_M$. Let $\chi : {}_\pi\mathrm{Res}\, M \to \kappa^*$ be the $(A \ltimes M)$-function given by ${}_{\langle a,m\rangle}\chi(m') = \langle a, m+m'\rangle$.

Theorem 3.1. *$\langle \chi, A \ltimes M, \pi_P \rangle$ is an extension in $\mathbf{V}$ of A by M.*

Proof. Each ${}_{\langle a,m\rangle}\chi$ is clearly one-one and onto, and χ is an homomorphism of pointed $(A\ltimes M)$-overalgebras because ${}_{\langle a,m\rangle}\chi(0) = \langle a, m\rangle = {}_{\langle a,m\rangle}*$, and for each $v \in \mathrm{Clo}_n\, \mathbf{V}$, each $\langle \mathbf{a}, \mathbf{m}\rangle = \langle\langle a_1, m_1\rangle, \ldots, \langle a_n, m_n\rangle\rangle \in (A \ltimes M)^n$, and each $\mathbf{m}' = \langle m'_1, \ldots, m'_n\rangle \in {}_{\langle \mathbf{a},\mathbf{m}\rangle}({}_\pi\mathrm{Res}\, M)$, we have

$$\begin{aligned}
{}_{v\langle \mathbf{a},\mathbf{m}\rangle}\chi(v^{{}_\pi\mathrm{Res}\, M}_{\langle \mathbf{a},\mathbf{m}\rangle}(\mathbf{m}')) &= {}_{v(\langle \mathbf{a},\mathbf{m}\rangle)}\chi(v^M_{\mathbf{a}}(\mathbf{m}')) \\
&= \langle v(\mathbf{a}), v^M_{\mathbf{a}}(\mathbf{m}) + v^M_{\mathbf{a}}(\mathbf{m}')\rangle \\
&= v^{A\ltimes M}(\langle a_1, m_1 + m'_1\rangle, \ldots, \langle a_n, m_n + m'_n\rangle) \\
&= v^{\kappa^*}_{\langle \mathbf{a},\mathbf{m}\rangle}(\langle a_1, m_1 + m'_1\rangle, \ldots, \langle a_n, m_n + m'_n\rangle) \\
&= v^{\kappa^*}_{\langle \mathbf{a},\mathbf{m}\rangle}({}_{\langle a_1,m_1\rangle}\chi(m'_1), \ldots, {}_{\langle a_n,m_n\rangle}\chi(m'_n)),
\end{aligned}$$

□

In this example, the onto homomorphism $\pi = \pi_M$ is split by the homomorphism ι_M. We say that an abelian extension $\langle \chi, E, \pi \rangle$ is *split* if π splits.

In general, given an extension $\langle \chi, E, \pi \rangle$ in $\mathbf{V}$ of A by M, the pointed E-overalgebra $\kappa^* = (\ker \pi)^*$ is isomorphic to the abelian group overalgebra ${}_\pi\mathrm{Res}\, M$, and so is itself an abelian group overalgebra with abelian group operations as given in theorem 1.1.

A lemma using the properties of the difference term. The following lemma will be very useful in proving properties of abelian extensions:

Lemma 3.2. *If $\langle \chi, E, \pi \rangle$ is an abelian extension in $\mathbf{V}$ of A by M, and e, e', $e'' \in E$ are such that $\pi(e) = \pi(e') = \pi(e'')$, then*

$$ {}_e\chi^{-1}(e') + {}_{e'}\chi^{-1}(e'') = {}_e\chi^{-1}(e''). $$

Proof. Let $\kappa = \ker \pi$, and let $a = \pi(e)$. By the properties of d and theorem 1.1, we have

$$\begin{aligned} {}_{e'}\chi^{-1}(e'') &= d^M_{\langle a,a,a\rangle}({}_{e'}\chi^{-1}(e''), {}_a0, {}_a0) \\ &= d^{\pi\mathrm{Res}\, M}_{\langle e',e',e\rangle}({}_{e'}\chi^{-1}(e''), {}_{e'}\chi^{-1}({}_{e'}0), {}_e\chi^{-1}({}_e0)) \\ &= {}_e\chi^{-1}(d^{\kappa^*}_{\langle e',e',e\rangle}(e'', {}_{e'}0, {}_e0)) \\ &= {}_e\chi^{-1}(d^E(e'', e', e)) \\ &= {}_e\chi^{-1}(d^{\kappa^*}_{\langle e,e,e\rangle}(e'', e', {}_e0)) \\ &= {}_e\chi^{-1}(e'') - {}_e\chi^{-1}(e'). \end{aligned}$$

□

Sections of extensions. Let $\mathcal{E} = \langle \chi, E, \pi \rangle$ be an abelian extension in $\mathbf{V}$ of A by M. Recall that we say that $\mathcal{E} = \langle \chi, E, \pi \rangle$ splits if there is a homomorphism $\sigma : A \to E$ right inverse to π. Not every abelian extension of A by M splits, as we know, because not every onto homomorphism has a right inverse. (We do know that there is always at least one split extension, given previously.) At any rate, by the axiom of choice, there is always a function σ such that $\pi\sigma = 1_A$, whether or not the extension $\mathcal{E}$ splits.

A function $\sigma : A \to E$, not necessarily a homomorphism, such that $\pi\sigma = 1_A$, will be called a *section* of π (or, of the extension $\mathcal{E}$).

***E* and *M*.** For each $e \in E$, ${}_e\chi^{-1} : \pi^{-1}(\pi(e)) \to {}_{\pi(a)}M$ is a one-one and onto function. A section σ chooses one representative $\sigma(a)$ for each κ-equivalence class $\pi^{-1}(a)$, and allows us to select functions ${}_{\sigma(a)}\chi^{-1}$ which provide an isomorphism of pointed A-sets between $[\![E, \pi, \sigma]\!]$ and M. We will use this pointed A-function extensively in what follows.

Some *A*-sets connected with abelian extensions. Let $\mathbf{V}$ be a variety of algebras of type Ω, and let A be an algebra in $\mathbf{V}$. We will define some A-sets that will be useful in the study of abelian extensions of A.

Let $X^0_{\mathbf{V}}(A)$, or simply X^0, denote the A-set $[\![A, 1_A]\!]$, that is, the underlying set of A, viewed as a set over A. We will refer to the element $a \in {}_aX^0$, for any $a \in A$, by $[a]$. Let $X^1_{\mathbf{V}}(A)$, or simply X^1, denote the A-set $[\![S^1, h]\!]$, where S^1 is the set of of pairs (written as follows)

$[v;\mathbf{a}]$, where $v \in \mathrm{Clo}_n \mathbf{V}$ for some n, and $\mathbf{a} \in A^n$, and $h[v;\mathbf{a}] = v(\mathbf{a})$. Let $X^2_{\mathbf{V}}(A)$, or simply X^2, denote the A-set $[\![S^2, k]\!]$, where S^2 is the set of triples $[v', \mathbf{v}; \mathbf{a}]$, where $\mathbf{a}$ is an element of A^n for some n, $\mathbf{v}$ is an n'-tuple of elements of $\mathrm{Clo}_n \mathbf{V}$, for some n', and $v' \in \mathrm{Clo}_{n'} \mathbf{V}$, and where $k[v', \mathbf{v}; \mathbf{a}] = v'(\mathbf{v}(\mathbf{a}))$.

Note that if $n = 0$, 1, or 2, then X^n satisfies the property that if $a \neq a'$, the sets ${}_aX^n$ and ${}_{a'}X^n$ are disjoint. We will often take advantage of this property in formulas which involve A-functions with domain X^n, by dropping the subscript a. Thus, if $x \in {}_aX^n$ and ϕ is an A-function from X^n to another A-set, we will write ${}_a\phi(x)$ simply as $\phi(x)$. This is unambiguous because x determines a.

$\delta^{\mathcal{E}}_{\sigma,\sigma'}$. As a first example of an A-function with domain one of the A-sets just defined, we define $\delta^{\mathcal{E}}_{\sigma,\sigma'}$ to be the A-function defined by

$$\delta^{\mathcal{E}}_{\sigma,\sigma'}([a]) = {}_{\sigma(a)}\chi^{-1}(\sigma'(a)),$$

where $\mathcal{E}$ is an extension and σ, σ' are two sections for $\mathcal{E}$.

Lemma 3.3. *If $\mathcal{E}$ is an abelian extension and σ is a section for $\mathcal{E}$, then $\delta^{\mathcal{E}}_{\sigma,\sigma} \equiv 0$.*

Proof. $\sigma(a) = {}_{\sigma(a)}0^{\kappa^*}$. Thus, $\delta^{\mathcal{E}}_{\sigma,\sigma} = {}_{\sigma(a)}\chi^{-1}(\sigma(a)) = {}_{\sigma(a)}\chi^{-1}(0) = 0$. □

4. Factor Sets

The factor set of an extension relative to a section. Let $\mathcal{E} = \langle \chi, E, \pi \rangle$ be an abelian extension of A by M, and σ a section. The functions ${}_{\sigma(a)}\chi^{-1}$, for $a \in A$, allow us to express the structure of E in terms of M. To begin we "measure the failure of σ to be a homomorphism" by defining, for each $[v;\mathbf{a}] \in X^1_{\mathbf{V}}(A)$, the element

$$\begin{aligned} \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}] &= {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma(\mathbf{a}))) - {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(\sigma(v(\mathbf{a}))) \\ &= {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma(\mathbf{a}))). \end{aligned}$$

It is easy to see that $\mathrm{f}^{\mathcal{E},\sigma}$ is an A-function from $X^1_{\mathbf{V}}(A)$ to M. We call $\mathrm{f}^{\mathcal{E},\sigma}$ the *factor set for $\mathcal{E}$, with respect to the section σ.*

Theorem 4.1. *If σ is a splitting for $\mathcal{E}$, then $\mathrm{f}^{\mathcal{E},\sigma} \equiv 0$.*

Proof. In that case, $\mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}] = {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma(\mathbf{a}))) = {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(\sigma(v(\mathbf{a}))) = {}_{v(\mathbf{a})}0$ for all v and $\mathbf{a}$. □

Theorem 4.2. *Together with σ, the factor set $\mathrm{f}^{\mathcal{E},\sigma}$ determines the algebra structure of E.*

Proof. Let $v \in \mathrm{Clo}_n \mathbf{V}$, $\mathbf{e} \in E^n$, and $\mathbf{a} = \pi(\mathbf{e})$. We have

$$\begin{aligned} {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\mathbf{e})) &= {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v^{\kappa^*}_{\sigma(\mathbf{a})}(\mathbf{e})) \\ &= {}_{v(\sigma(\mathbf{a}))}\chi^{-1}(v^{\kappa^*}_{\sigma(\mathbf{a})}(\mathbf{e})) + {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma(\mathbf{a}))) \\ &= v^{\pi \operatorname{Res} M}_{\sigma(\mathbf{a})}({}_{\sigma(\mathbf{a})}\chi^{-1}(\mathbf{e})) + \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}] \\ &= v^{M}_{\mathbf{a}}({}_{\sigma(\mathbf{a})}\chi^{-1}(\mathbf{e})) + \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}]. \end{aligned} \tag{4.1}$$

Thus, if the section σ is given, v^E and $\mathrm{f}^{\mathcal{E},\sigma}$ determine each other, because the mappings ${}_{\sigma(a)}\chi^{-1}$ together make up an isomorphism of A-sets from $[\![E,\pi]\!]$ to M. $\square$

Abstract factor sets. Because E is an algebra in $\mathbf{V}$, sending each $v \in \operatorname{Clo}_n \mathbf{V}$ to the operation v^E, for all n, is a clone homomorphism from $\operatorname{Clo}\mathbf{V}$ to the clone $\operatorname{Clo}U(E)$ of all finitary operations on the set $U(E)$. This means that for each $\mathbf{v} \in (\operatorname{Clo}_n \mathbf{V})^{n'}$, and each $v' \in \operatorname{Clo}_{n'} \mathbf{V}$, we have $v'^E\mathbf{v}^E = (v'\mathbf{v})^E$, where the clone composition on the left takes place in $\operatorname{Clo}U(E)$, and that on the right takes place in $\operatorname{Clo}\mathbf{V}$. Using equation (1) above, we obtain, for all $\mathbf{a} \in A^n$,

$$\begin{aligned}
\mathrm{f}^{\mathcal{E},\sigma}[v'\mathbf{v};\mathbf{a}] &= {}_{\sigma(v'(\mathbf{v}(\mathbf{a})))}\chi^{-1}(v'(\mathbf{v}(\sigma(\mathbf{a})))) \\
&= v'^M_{\mathbf{v}(\mathbf{a})}({}_{\sigma(\mathbf{v}(\mathbf{a}))}\chi^{-1}(\mathbf{v}(\sigma(\mathbf{a}))) + \mathrm{f}^{\mathcal{E},\sigma}[v';\mathbf{v}(\mathbf{a})] \\
&= v'^M_{\mathbf{v}(\mathbf{a})}\mathbf{v}^M_{\mathbf{a}}({}_{\sigma(\mathbf{a})}\chi^{-1}(\sigma(\mathbf{a}))) + v'^M_{\mathbf{v}(\mathbf{a})}\mathrm{f}^{\mathcal{E},\sigma}[\mathbf{v};\mathbf{a}] + \mathrm{f}^{\mathcal{E},\sigma}[v';\mathbf{v}(\mathbf{a})] \\
&= v'^M_{\mathbf{v}(\mathbf{a})}\mathrm{f}^{\mathcal{E},\sigma}[\mathbf{v};\mathbf{a}] + \mathrm{f}^{\mathcal{E},\sigma}[v';\mathbf{v}(\mathbf{a})],
\end{aligned}$$

for all such $\mathbf{v}$, v', and $\mathbf{a}$, where $\mathbf{v}(\mathbf{a})$ stands for $\langle v_1(\mathbf{a}), \dots, v_{n'}(\mathbf{a})\rangle$. We will call an A-function $\mathrm{f} : X^1_{\mathbf{V}}(A) \to M$ satisfying the family of equations

$$\mathrm{f}[v'\mathbf{v};\mathbf{a}] = v'^M_{\mathbf{v}(\mathbf{a})}\mathrm{f}[\mathbf{v};\mathbf{a}] + \mathrm{f}[v';\mathbf{v}(\mathbf{a})],$$

for each $[v',\mathbf{v};\mathbf{a}] \in X^2_{\mathbf{V}}(A)$, a *factor set for A and M*.

Let $A \in \mathbf{V}$ and let $M \in \mathbf{Ab}[A,\mathbf{V}]$. Given an arbitrary A-function $\mathrm{f} : X^1_{\mathbf{V}}(A) \to M$, we define an A-function $\partial\mathrm{f} : X^2_{\mathbf{V}}(A) \to M$ by the equation

$$(\partial\mathrm{f})[v',\mathbf{v};\mathbf{a}] = v'^M_{\mathbf{v}(\mathbf{a})}\mathrm{f}[\mathbf{v};\mathbf{a}] - \mathrm{f}[v'\mathbf{v};\mathbf{a}] + \mathrm{f}[v';\mathbf{v}(\mathbf{a})]$$

where $\mathbf{v}(\mathbf{a})$ is the n'-tuple $\langle v_1(\mathbf{a}), \dots, v_{n'}(\mathbf{a})\rangle$ and $[\mathbf{v};\mathbf{a}]$ is the n'-tuple $\langle [v_1;\mathbf{a}], \dots, [v_{n'};\mathbf{a}]\rangle$.

Theorem 4.3. *An A-function $\mathrm{f} : X^1_{\mathbf{V}}(A) \to M$ is a factor set for A and M iff $\partial\mathrm{f}$ is identically zero, i.e., if for each $[v',\mathbf{v};\mathbf{a}] \in X^2_{\mathbf{V}}(A)$, $(\partial\mathrm{f})[v',\mathbf{v};\mathbf{a}] = {}_{v'(\mathbf{v}(\mathbf{a}))}0$.*

Abstract factor sets and abelian extensions.

Theorem 4.4. *Every factor set for A and M arises as $\mathrm{f}^{\mathcal{E},\sigma}$ for some extension $\mathcal{E}$ of A by M and some section σ of that extension.*

Proof. Given a factor set f for A and M, we define $E = U(A \ltimes M)$, and for each $v \in \operatorname{Clo}_n(\mathbf{V})$, the n-ary operation $v^E : E^n \to E$, by the equation

$$v(\langle a_1, m_1\rangle, \dots, \langle a_n, m_n\rangle) = \langle v(\mathbf{a}), v^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v;\mathbf{a}]\rangle.$$

The fact that f is a factor set makes the mapping $v \to v^E$ a clone homomorphism, i.e., makes E an algebra of $\mathbf{V}$. For, let $e_i = \langle a_i, m_i\rangle$ for $i = 1, \dots, n$, let $\mathbf{v}$ be an n'-tuple of elements of $\operatorname{Clo}_n \mathbf{V}$, and let $v' \in \operatorname{Clo}_{n'} \mathbf{V}$. We have

$$\begin{aligned}
(v'\mathbf{v})(\mathbf{e}) &= \langle (v'\mathbf{v})(\mathbf{a}), (v'\mathbf{v})^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v'\mathbf{v};\mathbf{a}]\rangle \\
&= \langle (v'\mathbf{v})(\mathbf{a}), (v'\mathbf{v})^M_{\mathbf{a}}(\mathbf{m}) + v'^M_{\mathbf{v}(\mathbf{a})}\mathrm{f}[\mathbf{v};\mathbf{a}] + \mathrm{f}[v';\mathbf{v}(\mathbf{a})]\rangle,
\end{aligned}$$

while

$$\begin{aligned} v'(\mathbf{v}(\mathbf{e})) &= v'(\langle v_1(\mathbf{a}), (v_1)^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v_1;\mathbf{a}]\rangle, \dots \langle v_{n'}(\mathbf{a}), (v_{n'})^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v_{n'};\mathbf{a}]\rangle) \\ &= \langle v'(\mathbf{v}(\mathbf{a})), v'^M_{\mathbf{v}(\mathbf{a})}\mathbf{v}^M_{\mathbf{a}}(\mathbf{m}) + v'^M_{\mathbf{v}(\mathbf{a})}\mathrm{f}[\mathbf{v};\mathbf{a}] + \mathrm{f}[v';\mathbf{v}(\mathbf{a})]\rangle, \end{aligned}$$

and the desired equality of these two elements follows from the analogous facts for A and M.

Note $\pi_M : E \to A$ is a homomorphism with respect to the algebra E just defined. Let us denote $\ker \pi_M$ by κ, and ${}_{\pi_M}\mathrm{Res}\, M$ by $\bar{M}$. We have $\langle a, m\rangle\ \kappa\ \langle a', m'\rangle$ iff $a = a'$. We define $\chi : \bar{M} \to \kappa^*$ by the equation

$$_{\langle a,m\rangle}\chi(m') = \langle a, m' + m\rangle \in {}_{\langle a,m\rangle}\kappa^*.$$

Each $_{\langle a,m\rangle}\chi$ preserves the distinguished element because

$$_{\langle a,m\rangle}\chi({}_{\langle a,m\rangle}0) = {}_{\langle a,m\rangle}\chi({}_a0) = \langle a, m\rangle = {}_{\langle a,m\rangle}0 \in {}_{\langle a,m\rangle}\kappa^*.$$

Also, χ preserves the E-operations because for each $v \in \mathrm{Clo}_n(\mathbf{V})$, $e_i = \langle a_i, m_i\rangle \in E$ for $i = 1, \dots, n$, and $m_i \in {}_{e_i}\bar{M}$ for $i = 1, \dots, n$, we have

$$\begin{aligned} v^{\kappa^*}_{\mathbf{e}}({}_{\mathbf{e}}\chi(\mathbf{m}')) &= v^{\kappa^*}_{\mathbf{e}}(\langle a_1, m'_1 + m_1\rangle, \dots, \langle a_n, m'_n + m_n\rangle) \\ &= v^E(\langle a_1, m'_1 + m_1\rangle, \dots, \langle a_n, m'_n + m_n\rangle) \\ &= \langle v(\mathbf{a}), v^M_{\mathbf{a}}(\mathbf{m}') + v^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v;\mathbf{a}]\rangle, \end{aligned}$$

while on the other hand,

$$\begin{aligned} {}_{v(\mathbf{e})}\chi(v^{\bar{M}}_{\mathbf{e}}(\mathbf{m}')) &= {}_{\langle v(\mathbf{a}), v^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v;\mathbf{a}]\rangle}\chi(v^M_{\mathbf{a}}(\mathbf{m}')) \\ &= \langle v(\mathbf{a}), v^M_{\mathbf{a}}(\mathbf{m}') + v^M_{\mathbf{a}}(\mathbf{m}) + \mathrm{f}[v;\mathbf{a}]\rangle. \end{aligned}$$

Thus, χ is a homomorphism of pointed E-overalgebras, and it is clear that each $_e\chi$ is 1-1 and onto. It follows that χ is an isomorphism of pointed E-overalgebras.

Now, let $\sigma = \iota_M$ (so that $\sigma(a) = \langle a, {}_a0\rangle$) and let us show that the factor set we obtain is f. We have

$$_{\sigma(\pi_M\langle a,m\rangle)}\chi^{-1}\langle a, m\rangle = {}_{\langle a, {}_a0\rangle}\chi^{-1}\langle a, m\rangle,$$

so that the corresponding factor set is given by

$$\begin{aligned} \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}] &= {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma(\mathbf{a}))) \\ &= {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\langle a_1, {}_{a_1}0\rangle, \dots, \langle a_n, {}_{a_n}0\rangle)) \\ &= {}_{\langle v(\mathbf{a}), {}_{v(\mathbf{a}}0\rangle}\chi^{-1}\langle v(\mathbf{a}), \mathrm{f}[v;\mathbf{a}]\rangle \\ &= \mathrm{f}[v;\mathbf{a}], \end{aligned}$$

since by definition,

$$_{\langle v(\mathbf{a}), {}_{v(\mathbf{a})}0\rangle}\chi(\mathrm{f}[v;\mathbf{a}]) = \langle v(\mathbf{a}), \mathrm{f}[v;\mathbf{a}]\rangle.$$

$\square$

Effect of choice of section on the corresponding factor set. Let us see how factor sets for the same extension $\mathcal{E} = \langle \chi, \mathbf{E}, \pi \rangle$ in $\mathbf{V}$ of A by M differ when they are derived from different sections σ and σ'. First, a definition:

Let $\delta : X^0_{\mathbf{V}}(A) \to M$ be an A-function. We define an A-function $\partial\delta : X^1_{\mathbf{V}}(A) \to M$ by the equation

$$(\partial\delta)[v; \mathbf{a}] = v^M_{\mathbf{a}}(\delta[\mathbf{a}]) - \delta[v(\mathbf{a})],$$

for each $[v; \mathbf{a}] \in X^1_{\mathbf{V}}(A)$.

Theorem 4.5. *Under these assumptions, we have*

$$\mathrm{f}^{\mathcal{E},\sigma'} - \mathrm{f}^{\mathcal{E},\sigma} = \partial\delta^{\mathcal{E}}_{\sigma,\sigma'}.$$

Proof. Using equation (1) for $\mathbf{e} = \sigma'(\mathbf{a})$ to expand $\mathrm{f}^{\mathcal{E},\sigma}[v; \mathbf{a}]$, we have

$$\mathrm{f}^{\mathcal{E},\sigma'}[v; \mathbf{a}] - \mathrm{f}^{\mathcal{E},\sigma}[v; \mathbf{a}] = {}_{\sigma'(v(\mathbf{a}))}\chi^{-1}(v(\sigma'(\mathbf{a}))) + v^M_{\mathbf{a}}({}_{\sigma(\mathbf{a})}\chi^{-1}(\sigma'(\mathbf{a}))) - {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma'(\mathbf{a}))).$$

But,

$${}_{\sigma'(v(\mathbf{a}))}\chi^{-1}(v(\sigma'(\mathbf{a}))) = {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma'(\mathbf{a}))) - {}_{\sigma(v(a))}\chi^{-1}(\sigma'(v(\mathbf{a})))$$

by lemma 3.1; combining these two results, we obtain

$$\begin{aligned} \mathrm{f}^{\mathcal{E},\sigma'}[v; \mathbf{a}] - \mathrm{f}^{\mathcal{E},\sigma}[v; \mathbf{a}] &= v^M_{\mathbf{a}}({}_{\sigma(\mathbf{a})}\chi^{-1}(\sigma'(\mathbf{a}))) - {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(\sigma'(v(\mathbf{a}))) \\ &= v^M_{\mathbf{a}}(\delta^{\mathcal{E}}_{\sigma,\sigma'}[\mathbf{a}]) - \delta^{\mathcal{E}}_{\sigma,\sigma'}[v(\mathbf{a})] \\ &= (\partial\delta^{\mathcal{E}}_{\sigma,\sigma'})[v; \mathbf{a}], \end{aligned}$$

as was to be proved. □

5. Equivalence of Abelian Extensions

Let E and $\tilde{E}$ be algebras of $\mathbf{V}$, a variety of algebras of type Ω. Let $\gamma : E \to \tilde{E}$ be a homomorphism, and let α and $\tilde{\alpha}$ be congruences of E and $\tilde{E}$, respectively, such that $\gamma(\alpha) \subseteq \tilde{\alpha}$. Then we define an E-function, $\gamma^* : \alpha^* \to {}_{\gamma}\mathrm{Res}(\tilde{\alpha}^*)$ by the equation

$${}_e\gamma^*(e') = \gamma(e').$$

Theorem 5.1. *γ^* is a homomorphism of pointed E-overalgebras.*

Equivalent extensions. Let $\mathcal{E} = \langle \chi, E, \pi \rangle$ and $\tilde{\mathcal{E}} = \langle \tilde{\chi}, \tilde{E}, \tilde{\pi} \rangle$ be abelian extensions in $\mathbf{V}$ of A by M, where $M \in \mathbf{Ab}[A, \mathbf{V}]$. We define an *equivalence of extensions* from $\mathcal{E}$ to $\tilde{\mathcal{E}}$ to be a homomorphism $\gamma : E \to \tilde{E}$, such that

(1) $\pi = \tilde{\pi}\gamma$, and
(2) $\gamma^*\chi = {}_{\gamma}\mathrm{Res}\,\tilde{\chi}$.

If $\mathcal{E}$ and $\hat{\mathcal{E}}$ are equivalent via an equivalence γ, we write $\gamma : \mathcal{E} \sim \hat{\mathcal{E}}$.

Bearing in mind that, because of condition (1), ${}_{\gamma}\mathrm{Res} \circ {}_{\tilde{\pi}}\mathrm{Res} = {}_{\pi}\mathrm{Res}$, we express these conditions in the following interrelated diagrams:

$$
\begin{array}{ccccccc}
M & {}_{\pi}\mathrm{Res}\, M & \xrightarrow{\chi} & \kappa^* & E & \xrightarrow{\pi} & A \\
\Big\| & \Big\| & & \Big\downarrow \gamma^* & \gamma \Big\downarrow & & \Big\| \\
M & {}_{\pi}\mathrm{Res}\, M & \xrightarrow[{}_{\gamma}\mathrm{Res}\, \tilde{\chi}]{} & {}_{\gamma}\mathrm{Res}(\tilde{\kappa}^*) & \tilde{E} & \xrightarrow[\tilde{\pi}]{} & A \\
 & & & & & & \\
 & {}_{\tilde{\pi}}\mathrm{Res}\, M & \xrightarrow[\tilde{\chi}]{} & \tilde{\kappa}^* & & &
\end{array}
$$

where $\kappa = \ker \pi$ and $\tilde{\kappa} = \ker \tilde{\pi}$.

Theorem 5.2. *Equivalence of extensions is an equivalence relation on extensions in* $\mathbf{V}$ *of* A *by* M.

Proof. If $\mathcal{E} = \langle \chi, E, \pi \rangle$, $\tilde{\mathcal{E}} = \langle \tilde{\chi}, \tilde{E}, \tilde{\pi} \rangle$, and $\bar{\mathcal{E}} = \langle \bar{\chi}, \bar{E}, \bar{\pi} \rangle$ are extensions in $\mathbf{V}$ of A by M, and $\gamma_1 : \mathcal{E} \sim \tilde{\mathcal{E}}$, $\gamma_2 : \tilde{\mathcal{E}} \sim \bar{\mathcal{E}}$ are equivalences, then we first observe that

$$(\gamma_2\gamma_1)^* = ({}_{\gamma_1}\mathrm{Res}\, \gamma_2^*)\gamma_1^*.$$

Then, we have

$$
\begin{aligned}
(\gamma_2\gamma_1)^*\chi &= ({}_{\gamma_1}\mathrm{Res}\, \gamma_2^*)\gamma_1^*\chi \\
&= ({}_{\gamma_1}\mathrm{Res}\, \gamma_2^*)\, {}_{\gamma_1}\mathrm{Res}\, \tilde{\chi} \\
&= {}_{\gamma_1}\mathrm{Res}(\gamma_2^*\tilde{\chi}) \\
&= {}_{\gamma_1}\mathrm{Res}({}_{\gamma_2}\mathrm{Res}\, \bar{\chi}) \\
&= {}_{\gamma_2\gamma_1}\mathrm{Res}\, \bar{\chi};
\end{aligned}
$$

we also have $\bar{\pi}\gamma_2\gamma_1 = \pi$, whence $\gamma_2\gamma_1 : \mathcal{E} \sim \bar{\mathcal{E}}$. □

Lemma 5.3. *If* γ *is an equivalence from* $\mathcal{E}$ *to* $\tilde{\mathcal{E}}$, *then* γ *is an isomorphism, and is the unique equivalence from* $\mathcal{E}$ *to* $\tilde{\mathcal{E}}$.

Proof. χ and ${}_{\gamma}\mathrm{Res}\, \chi$ are isomorphisms, whence γ^* is also by condition (2). Thus, γ maps each κ-class onto a $\tilde{\kappa}$-class, in a one-one fashion. Since each $\tilde{\kappa}$-class is the image of a unique κ-class by condition (1), these mappings paste together to give γ as an isomorphism.

γ is unique, because it is determined by γ^*, which is determined by condition (2). □

We denote by $E_{\mathbf{V}}(A, M)$ the set of equivalence classes of extensions in $\mathbf{V}$ of A by M. It is easy to see that if $\mathcal{E} \sim \tilde{\mathcal{E}}$, then $\mathcal{E}$ splits iff $\tilde{\mathcal{E}}$ splits. Thus, the split extensions of A by M form one or more equivalence classes. We will see below, in corollary 5.6, that all split extensions are equivalent.

Equivalence and factor sets. We wish to relate factor sets and equivalence.

Lemma 5.4. *If $\gamma : \mathcal{E} \sim \tilde{\mathcal{E}}$, then given a section σ of $\mathcal{E}$, we have $\mathrm{f}^{\mathcal{E},\sigma} = \mathrm{f}^{\tilde{\mathcal{E}},\gamma\sigma}$.*

Proof. For each $[v;\mathbf{a}]$, we have

$$\begin{aligned}
\mathrm{f}^{\tilde{\mathcal{E}},\gamma\sigma}[v;\mathbf{a}] &= {}_{\gamma(\sigma(v(\mathbf{a})))}\tilde{\chi}^{-1}(v(\gamma(\sigma(\mathbf{a})))) \\
&= {}_{\sigma(v(\mathbf{a}))}({}_{\gamma}\mathrm{Res}\,\tilde{\chi})^{-1}({}_{\sigma(v(\mathbf{a}))}\gamma^*(v(\sigma(\mathbf{a})))) \\
&= {}_{\sigma(v(\mathbf{a}))}\chi^{-1}(v(\sigma(\mathbf{a}))) \\
&= \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}].
\end{aligned}$$

□

Theorem 5.5. *Let M be an A-module. Let $\mathcal{E} = \langle \chi, E, \pi \rangle$ and $\tilde{\mathcal{E}} = \langle \tilde{\chi}, \tilde{E}, \tilde{\pi} \rangle$ be two abelian extensions in $\mathbf{V}$ of A by M, with sections σ and τ, respectively. Then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent if and only if the factor sets $\mathrm{f}^{\mathcal{E},\sigma}$, $\mathrm{f}^{\tilde{\mathcal{E}},\tau} : X^1_{\mathbf{V}}(A) \to M$ differ by a function of the form $\partial\delta$ where $\delta : X^0_{\mathbf{V}}(A) \to M$ is an A-function.*

Proof. Suppose $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent via an equivalence γ. By lemma 5.4, $\mathrm{f}^{\mathcal{E},\sigma} = \mathrm{f}^{\tilde{\mathcal{E}},\gamma\sigma}$. On the other hand, $\mathrm{f}^{\tilde{\mathcal{E}},\gamma\sigma} - \mathrm{f}^{\tilde{\mathcal{E}},\tau} = \partial\delta^{\tilde{\mathcal{E}}}_{\tau,\gamma\sigma}$ by theorem 4.5. The desired result follows.

Conversely, if the factor set difference $\mathrm{f}^{\tilde{\mathcal{E}},\tau} - \mathrm{f}^{\mathcal{E},\sigma} = \partial\delta$ for $\delta : X^0_{\mathbf{V}}(A) \to M$ some A-function, then we can produce a new section $\hat{\tau}$ for $\tilde{\mathcal{E}}$, namely

$$\hat{\tau}(a) = {}_{\tau(a)}\tilde{\chi}(\delta[a]),$$

and we have $\mathrm{f}^{\tilde{\mathcal{E}},\hat{\tau}} = \mathrm{f}^{\mathcal{E},\sigma}$. For,

$$\begin{aligned}
\mathrm{f}^{\tilde{\mathcal{E}},\hat{\tau}}[v;\mathbf{a}] &= {}_{\hat{\tau}(v(\mathbf{a}))}\tilde{\chi}^{-1}(v(\hat{\tau}(\mathbf{a}))) \\
&= {}_{\hat{\tau}(v(\mathbf{a}))}\tilde{\chi}^{-1}(v({}_{\tau(\mathbf{a})}\tilde{\chi}(\delta[\mathbf{a}]))) \\
&= {}_{\tau(v(\mathbf{a}))}\tilde{\chi}^{-1}(v({}_{\tau(\mathbf{a})}\tilde{\chi}(\delta[\mathbf{a}]))) - {}_{\tau(v(\mathbf{a}))}\tilde{\chi}^{-1}(\hat{\tau}(v(\mathbf{a}))) \\
&= v^M_{\mathbf{a}}({}_{\tau(\mathbf{a})}\chi^{-1}({}_{\tau(\mathbf{a})}\chi(\delta[\mathbf{a}]))) + \mathrm{f}^{\tilde{\mathcal{E}},\tau}[v;\mathbf{a}] - {}_{\tau(v(\mathbf{a}))}\tilde{\chi}^{-1}({}_{\tau(v(\mathbf{a}))}\tilde{\chi}(\delta[v(\mathbf{a})])) \\
&= \mathrm{f}^{\tilde{\mathcal{E}},\tau}[v;\mathbf{a}] + v^M_{\mathbf{a}}(\delta[\mathbf{a}]) - \delta[v(\mathbf{a})] \\
&= \mathrm{f}^{\tilde{\mathcal{E}},\tau}[v;\mathbf{a}] + (\partial\delta)[v;\mathbf{a}] \\
&= \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}],
\end{aligned}$$

as desired.

Now we will construct an isomorphism $\gamma : E \to \tilde{E}$, which is an equivalence from $\mathcal{E}$ to $\tilde{\mathcal{E}}$. We define the one-one and onto function $\gamma : E \to \tilde{E}$ by

$$\gamma(e) = {}_{\hat{\tau}(\pi(e))}\tilde{\chi}({}_{\sigma(\pi(e))}\chi^{-1}(e)).$$

We have $\pi = \tilde{\pi}\gamma$, because γ maps each $\pi^{-1}(a)$ to $\tilde{\pi}^{-1}(a)$, and we have $\gamma\sigma = \hat{\tau}$, because of lemma 3.3. If we are given $\mathbf{e} \in E^n$, then letting $\mathbf{a} = \pi(\mathbf{e}) = \tilde{\pi}(\gamma(\mathbf{e}))$, we have for each

$v \in \mathrm{Clo}_n(\mathbf{V})$,

$$\begin{aligned}\gamma(v(\mathbf{e})) &= {}_{\hat\tau(v(\mathbf{a}))}\tilde\chi(v^M_{\mathbf{a}}({}_{\sigma(\mathbf{a})}\chi^{-1}(\mathbf{e})) + \mathrm{f}^{\mathcal{E},\sigma}[v;\mathbf{a}]) \\ &= {}_{\hat\tau(v(\mathbf{a}))}\tilde\chi(v^M_{\mathbf{a}}({}_{\hat\tau(\mathbf{a})}\tilde\chi^{-1}(\gamma(\mathbf{e}))) + \mathrm{f}^{\tilde{\mathcal{E}},\hat\tau}[v;\mathbf{a}]) \\ &= {}_{\hat\tau(v(\mathbf{a}))}\tilde\chi({}_{\hat\tau(v(\mathbf{a}))}\tilde\chi^{-1}(v(\gamma(\mathbf{e}))) \\ &= v(\gamma(\mathbf{e})),\end{aligned}$$

whence the 1-1 and onto function γ is an isomorphism. Finally, for each $e \in E$, with $a = \pi(e)$, and all $m \in {}_aM$, we have

$$\begin{aligned}{}_e({}_\gamma\mathrm{Res}\,\tilde\chi)(m) &= {}_{\gamma(e)}\tilde\chi(m) \\ &= {}_{\hat\tau(a)}\tilde\chi({}_{\hat\tau(a)}\tilde\chi^{-1}({}_{\gamma(e)}\tilde\chi(m))) \\ &= {}_{\hat\tau(a)}\tilde\chi({}_{\hat\tau(a)}\tilde\chi^{-1}(\gamma(e)) + {}_{\gamma(e)}\tilde\chi^{-1}({}_{\gamma(e)}\tilde\chi(m))) \\ &= {}_{\hat\tau(a)}\tilde\chi({}_{\sigma(a)}\chi^{-1}(e) + m) \\ &= {}_{\hat\tau(a)}\tilde\chi({}_{\sigma(a)}\chi^{-1}(e) + {}_e\chi^{-1}({}_e\chi(m))) \\ &= {}_{\hat\tau(a)}\tilde\chi({}_{\sigma(a)}\chi^{-1}({}_e\chi(m))) \\ &= {}_e(\gamma^*\chi)(m);\end{aligned}$$

thus, $\gamma^*\chi = {}_\gamma\mathrm{Res}\,\tilde\chi$, so that condition (2) is satisfied, and $\gamma : \mathcal{E} \sim \tilde{\mathcal{E}}$. □

Corollary 5.6. *The split extensions of A by M form a single equivalence class.*

Proof. If $\mathcal{E}$ is an extension in $\mathbf{V}$ of A by M, and σ is a splitting, then $\mathrm{f}^{\mathcal{E},\sigma} \equiv 0$ by theorem 4.1. If τ is another section of $\mathcal{E}$, then $\mathrm{f}^{\mathcal{E},\tau} = \partial\delta$ for some A-function $\delta : X^0 \to M$, by theorem 4.5. Thus, every factor set of a split extension has the form $\partial\delta$. It follows by the theorem that all split extensions are equivalent. □

6. $\mathbf{E}_{\mathbf{V}}(A, M)$ as a Homology Object

A fragment of a cochain complex. We previously introduced A-sets $X^0 = X^0_{\mathbf{V}}(A)$, $X^1 = X^1_{\mathbf{V}}(A)$, and $X^2 = X^2_{\mathbf{V}}(A)$, and, with the additional data of an object $M \in \mathbf{Ab}[A, \mathbf{V}]$, abelian group homomorphisms ∂, so that the diagram

$$A\text{-}\mathbf{Set}(X^0, M) \xrightarrow{\partial} A\text{-}\mathbf{Set}(X^1, M) \xrightarrow{\partial} A\text{-}\mathbf{Set}(X^2, M)$$

is a fragment of a cochain complex in the abelian category $\mathbf{Ab}$.

The cohomology group. An A-function $\mathrm{f} : X^1 \to M$ is a factor set for A and M iff $\partial\mathrm{f} = 0$, by definition. We showed (theorem 4.4) that each such abstract factor set is a factor set $\mathrm{f}^{\mathcal{E},\sigma}$ for some extension $\mathcal{E}$ and section σ. On the other hand, two factor sets f, f′ correspond to equivalent extensions iff $\mathrm{f} - \mathrm{f}' = \partial\delta$ for some A-function $\delta : X^0 \to M$. It follows that

Theorem 6.1. *The equivalence classes of extensions of A by M, i.e., the elements of the set $\mathbf{E}_{\mathbf{V}}(A, M)$, correspond to the elements of the cohomology group of the above fragment of a cochain complex in the abelian category* $\mathbf{Ab}$.

The equivalence class of split extensions.

Theorem 6.2. *The class of split extensions is the zero element of* $\mathbf{E_V}(A, M)$.

Proof. Choosing a split extension $\mathcal{E}$, and a splitting σ as the section, we obtain $\mathrm{f}^{\mathcal{E},\sigma} \equiv 0$, by theorem 4.1. □

$\mathbf{E_V}(A, M)$ **as an abelian group.** The correspondence of theorem 6.1 allows us to place the structure of an abelian group on the set $\mathbf{E_V}(A, M)$: If $\mathcal{E}_1$, $\mathcal{E}_2$ are represented by factor sets f_1 and f_2, then $[\mathcal{E}_1] + [\mathcal{E}_2]$ is represented by $\mathrm{f}_1 + \mathrm{f}_2$.

Theorem 6.3. *If* $\mathcal{E} = \langle \chi, E, \pi \rangle$ *is an extension of* A *by* M*, then* $-[\mathcal{E}] = [\langle -\chi, E, \pi \rangle]$.

Proof. $-\chi$ is also an isomorphism, and the formula for factor sets shows that

$$\mathrm{f}^{\langle -\chi, E, \pi \rangle, \sigma} = -\mathrm{f}^{\langle \chi, E, \pi \rangle, \sigma},$$

for any section σ. □

7. Compositions of Extensions and Homomorphisms

For this section, let $\mathcal{E} = \langle \chi, E, \pi \rangle$ be an abelian extension of A by M. We will describe ways of composing $\mathcal{E}$ with homomorphisms in $\mathbf{V}$ and $\mathbf{Ab}[A, \mathbf{V}]$. Conceptually, these composition operations are best seen as operations on the *extension class* $[\mathcal{E}]$, the equivalence class of extensions in $\mathbf{V}$ of A by M represented by $\mathcal{E}$. However, it is not hard to define the composition of an extension and an algebra homomorphism as a specific extension, and sometimes useful, so we will give the definition of that composition in that form.

The extension $\mathcal{E}g$ for a homomorphism $g : A' \to A$. Suppose that we have another algebra A' and a homomorphism $g : A' \to A$. We will show how to create from $\mathcal{E}$ an extension of A' by ${}_g\mathrm{Res}\, M$. The first step is to consider the fibered product E' of A' and E, with the associated homomorphisms which we label π' and γ:

$$\begin{array}{ccc} E' & \xrightarrow{\pi'} & A' \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle g.} \\ E & \xrightarrow[\pi]{} & A \end{array}$$

Lemma 7.1. *In this situation, if* $\kappa = \ker \pi$ *and* $\kappa' = \ker \pi'$*, then* $\gamma^* : \kappa'^* \to {}_\gamma\mathrm{Res}\, \kappa^*$ *is an isomorphism of pointed* E'*-overalgebras.*

Proof. If $\langle e, a' \rangle$ is an element of E', then the elements of ${}_{\langle e,a' \rangle}\kappa'^*$ are the elements of E' of the form $\langle e', a' \rangle$, and such a pair will belong to E' iff $\pi(e') = g(a')$. On the other hand, ${}_{\langle e,a' \rangle}({}_\gamma\mathrm{Res}\, \kappa^*) = {}_e\kappa^*$ is the set of e' such that $\pi(e) = \pi(e')$. However, we have $g(a') = \pi(e)$, because $\langle e, a' \rangle \in E'$. Thus, the mapping $\langle e', a' \rangle \mapsto e'$ is one-one and onto. Since this applies to each ${}_{\langle e,a' \rangle}\gamma^*$, γ^* is an isomorphism. □

Now, we define $\chi' = (\gamma^*)^{-1} \circ {}_{\gamma}\mathrm{Res}\,\chi$. Since γ^* and χ are isomorphisms, so is χ'. Also, we have $\pi\gamma = g\pi'$, so that $\chi' : {}_{\pi'}\mathrm{Res}({}_{g}\mathrm{Res}\,M) \cong \kappa'^*$. Thus, we obtain an extension $\mathcal{E}g = \langle \chi', E', \pi' \rangle$ of A' by ${}_{g}\mathrm{Res}\,M$. We will denote the extension class $[\mathcal{E}g]$ by $[\mathcal{E}]g$.

If σ is any section for $\mathcal{E}$, let $\sigma' : A' \to E'$ be defined by $\sigma'(a') = \langle \sigma(g(a')), a' \rangle$. Then, we have

Lemma 7.2. *If* $\mathrm{f} = \mathrm{f}^{\mathcal{E},\sigma}$*, then there is an extension* $\bar{\mathcal{E}} \in [\mathcal{E}]g$*, and a section* $\bar{\sigma}$*, such that or each* $[v; \mathbf{c}\,] \in X^1_{\mathbf{V}}(A')$*,* $\mathrm{f}^{\bar{\mathcal{E}},\bar{\sigma}}[v; \mathbf{c}\,] = \mathrm{f}[v; g(\mathbf{c}\,)]$*.*

Proof. Let $\bar{\mathcal{E}} = \mathcal{E}g$ and $\bar{\sigma} = \sigma'$ as defined above. We have

$$\begin{aligned} \mathrm{f}^{\mathcal{E}g,\sigma'}[v; \mathbf{c}\,] &= {}_{\sigma'(v(\mathbf{c}))}{\chi'}^{-1}(v(\sigma'(\mathbf{c}\,))) \\ &= {}_{\sigma'(v(\mathbf{c}))}(\gamma^{*-1} \circ {}_{\gamma}\mathrm{Res}\,\chi)^{-1}(v(\sigma'(\mathbf{c}\,)) \\ &= {}_{\sigma'(v(\mathbf{c}))}({}_{\gamma}\mathrm{Res}\,\chi^{-1})(\gamma^*(v(\sigma'(\mathbf{c}\,)))) \\ &= {}_{\sigma(v(g(\mathbf{c})))}\chi^{-1}(v(\sigma(g(\mathbf{c}\,)))) \\ &= \mathrm{f}^{\mathcal{E},\sigma}[v; g(\mathbf{c}\,)]. \end{aligned}$$

□

Corollary 7.3. *If in addition to* $\mathcal{E}$*,* A'*, and* g *we have an algebra* A'' *and homomorphism* $h : A'' \to A'$*, then* $\mathcal{E}(gh) \sim (\mathcal{E}g)h$*.*

Corollary 7.4. *The mapping* $[\mathcal{E}] \mapsto [\mathcal{E}]g$ *is a group homomorphism from* $\mathbf{E}_{\mathbf{V}}(A, M)$ *to the group* $\mathbf{E}_{\mathbf{V}}(A', {}_{g}\mathrm{Res}\,M)$*.*

The extension class $\dot{g}[\mathcal{E}]$ for a homomorphism $\dot{g} : M \to M'$. Suppose again that $\mathcal{E} = \langle \chi, E, \pi \rangle$ is given, and that M' is another object of $\mathbf{Ab}[A, \mathbf{V}]$, and $\dot{g} : M \to M'$ a homomorphism. $\mathcal{E}$ determines an extension class $[\mathcal{E}]$. We will specify an extension class $\dot{g}[\mathcal{E}]$ of A by M' by giving a representative factor set.

Let $\mathrm{f}^{\mathcal{E},\sigma}$ be a factor set for $\mathcal{E}$, where σ is some section. Since $\mathrm{f}^{\mathcal{E},\sigma}$ is an A-function from X^1 to M, and $\dot{g}$ is a homomorphism from M to M', we can compose them and obtain an A-function $\mathrm{f}' = \dot{g}\mathrm{f}^{\mathcal{E},\sigma} : X^1 \to M'$.

Theorem 7.5. f' *is a factor set for* A *and* M'*.*

Proof. We have

$$\begin{aligned} (\partial \mathrm{f}')[v', \mathbf{v}; \mathbf{a}] &= {v'}^{M'}_{\mathbf{v}(\mathbf{a})}\mathrm{f}'[\mathbf{v}; \mathbf{a}] - \mathrm{f}'[v'\mathbf{v}; \mathbf{a}] + \mathrm{f}'[v'; \mathbf{v}(\mathbf{a})] \\ &= \dot{g}{v'}^{M}_{\mathbf{v}(\mathbf{a})}\mathrm{f}[\mathbf{v}; \mathbf{a}] - \dot{g}\mathrm{f}[v'\mathbf{v}; \mathbf{a}] + \dot{g}\mathrm{f}[v'; \mathbf{v}(\mathbf{a})] \\ &= \dot{g}(\partial \mathrm{f})[v', \mathbf{v}; \mathbf{a}] \\ &= \dot{g}({}_{v'\mathbf{v}(\mathbf{a})}0^M) \\ &= {}_{v'\mathbf{v}(\mathbf{a})}0^{M'}. \end{aligned}$$

□

Thus, by the construction of theorem 4.4, f' is the factor set of some extension, which we shall for the moment denote by $\mathcal{E}^{\sigma,\dot{g}}$.

Theorem 7.6. *If σ' is another section of $\mathcal{E}$, then $\mathcal{E}^{\sigma,\dot{g}} \sim \mathcal{E}^{\sigma',\dot{g}}$. I.e., $[\mathcal{E}^{\sigma,\dot{g}}]$ does not depend on σ.*

Proof. $\mathrm{f}^{\mathcal{E},\sigma} - \mathrm{f}^{\mathcal{E},\sigma'} = \partial\delta$ for some A-function $\delta : X^0 \to M$. Then, $\dot{g}\mathrm{f}^{\mathcal{E},\sigma} - \dot{g}\mathrm{f}^{\mathcal{E},\sigma'} = \dot{g}(\partial\delta)$. However, $\dot{g}(\partial\delta) = \partial(\dot{g}\delta)$. For,

$$\begin{aligned} \dot{g}(\partial\delta)[v;\mathbf{a}] &= \dot{g}(v^M_{\mathbf{a}}(\delta[\mathbf{a}]) - \delta[v(\mathbf{a})]) \\ &= v^{M'}_{\mathbf{a}}(\dot{g}\delta[\mathbf{a}]) - \dot{g}\delta[v(\mathbf{a})] \\ &= \partial(\dot{g}\delta)[v;\mathbf{a}]; \end{aligned}$$

Thus, $\mathcal{E}^{\sigma,\dot{g}} \sim \mathcal{E}^{\sigma',\dot{g}}$ by theorem 5.5. □

We denote $[\mathcal{E}^{\sigma,\dot{g}}]$ by $\dot{g}[\mathcal{E}]$.

Theorem 7.7. *If M'' is another object of $\mathbf{Ab}[A,\mathbf{V}]$, and $\dot{h} : M' \to M''$ is another homomorphism, then $(\dot{h}\dot{g})[\mathcal{E}] = \dot{h}(\dot{g}[\mathcal{E}])$.*

Theorem 7.8. *The mapping $[\mathcal{E}] \mapsto \dot{g}[\mathcal{E}]$ is an abelian group homomorphism from $\mathbf{E}_{\mathbf{V}}(A,M)$ to $\mathbf{E}_{\mathbf{V}}(A,M')$.*

Proof. Follows from the definition of $\dot{g}$ being a homomorphism of abelian group objects in the category $\mathbf{Ov}[A,\mathbf{V}]$. □

Relationship of these two compositions.

Theorem 7.9. *If $\mathcal{E}$ is an extension of A by M, and g and $\dot{g}$ are given, then $(\dot{g}[\mathcal{E}])g = ({}_g\mathrm{Res}\,\dot{g})([\mathcal{E}]g)$.*

Proof. We will show that representative factor sets for these two extension classes are equal.

Let f be a factor set representing the homology class corresponding to the extension class of $\mathcal{E}$. Then $\dot{g}\mathrm{f}$ represents $\dot{g}[\mathcal{E}]$. A factor set f' for $(\dot{g}[\mathcal{E}])g$ can then be defined by

$$\mathrm{f}' : [v;\mathbf{c}\,] \mapsto (\dot{g}\mathrm{f})[v;g(\mathbf{c}\,)].$$

On the other hand, a factor set $\bar{\mathrm{f}}$ for $[\mathcal{E}]g$ can be defined by

$$\bar{\mathrm{f}} : [v;\mathbf{c}\,] \mapsto \mathrm{f}[v;g(\mathbf{c}\,)]$$

and then a factor set for $({}_g\mathrm{Res}\,\dot{g})([\mathcal{E}]g)$ can be given by $({}_f\mathrm{Res}\,\dot{g})\bar{\mathrm{f}}$. For each $[v;\mathbf{c}\,]$, we have

$$\begin{aligned} ({}_g\mathrm{Res}\,\dot{g})\bar{\mathrm{f}}[v;\mathbf{c}\,] &= {}_{g(v(\mathbf{c}\,))}\dot{g}(\mathrm{f}[v;g(\mathbf{c}\,)]) \\ &= \dot{g}\mathrm{f}[v;g(\mathbf{c}\,)] \\ &= \mathrm{f}'[v;\mathbf{c}\,]. \end{aligned}$$

□

8. The Addition Operation on $\mathbf{E_V}(Q, M)$

Now we will describe a method for constructing the result of adding a pair $\mathcal{E}_1$, $\mathcal{E}_2$ of extensions of A by M. This method closely parallels the method of adding two module extensions called the Baer sum. However, while the Baer sum uses only universal properties to construct an extension having the desired properties, we can do this only up to a point, and must complete the construction of a representative of $[\mathcal{E}_1]+[\mathcal{E}_2]$ by using factor sets and theorem 4.4.

Outer product of a finite tuple of extensions. Let $\mathcal{E}_i = \langle \chi_i, E_i, \pi_i\rangle$ be an extension of A_i by M_i for $i = 1, \dots, n$, where $A_i \in \mathbf{V}$ and $M_i \in \mathbf{Ab}[A_i\mathbf{V}]$ for all i. Define $E = \Pi_i E_i$ and $A = \Pi_i A_i$. Define $\pi : E \to A$ by the equation $\pi(\mathbf{e}) = \langle \pi_1(e_1), \dots, \pi_n(e_n)\rangle$. We will construct an extension $\mathcal{E} = \langle \chi, E, A\rangle$ of A by $\boxtimes_i M_i$ from the extensions $\mathcal{E}_i$.

If $\kappa_i = \ker \pi_i$ for all i, $\kappa = \ker \pi$, and $\pi_{E,i} : E \to E_i$ are the projections to the factors, then we have

Lemma 8.1. *κ^* is naturally isomorphic to $\Pi_i({}_{\pi_{E,i}}\mathrm{Res}\,\kappa_i^*)$.*

We define $\chi : {}_\pi\mathrm{Res}\, M \to \kappa^*$ by the equation

$$ {}_{\mathbf{e}}\chi(\mathbf{m}) = \langle {}_{e_1}\chi_1(m_1), \dots, {}_{e_n}\chi_n(m_n)\rangle, $$

and finally, we have

Theorem 8.2. *$\langle \chi, E, \pi\rangle$ is an extension of A by M.*

We denote this extension by $\boxtimes_i \mathcal{E}_i$.

A factor set of $\boxtimes_i \mathcal{E}_i$. Let σ_i be a section of $\mathcal{E}_i$ for each i, and let $\sigma : A \to E$ be the section of $\boxtimes_i \mathcal{E}_i$ defined by $\sigma(\mathbf{a}) = \langle \sigma_1(a_1), \dots, \sigma_n(a_n)\rangle$.

Theorem 8.3. *For each $[v; \langle \mathbf{a}_1, \dots, \mathbf{a}_\ell\rangle] \in X^1_{\mathbf{V}}(A)$, we have*

$$ \mathrm{f}^{\boxtimes_i\mathcal{E}_i,\sigma}[v; \langle \mathbf{a}_1, \dots, \mathbf{a}_\ell\rangle] = \langle \mathrm{f}^{\mathcal{E}_1,\sigma_1}[v; \langle a_{11}, \dots, a_{\ell 1}\rangle], \dots, \mathrm{f}^{\mathcal{E}_n,\sigma_n}[v; \langle a_{1n}, \dots, a_{\ell n}\rangle]\rangle. $$

Proof. We have

$$ \begin{aligned} &\mathrm{f}^{\boxtimes_i\mathcal{E}_i,\sigma}[v; \langle \mathbf{a}_1, \dots, \mathbf{a}_\ell\rangle] \\ &\quad = {}_{\sigma(v(\mathbf{a}_1,\dots,\mathbf{a}_\ell))}\chi^{-1}(v(\sigma(\mathbf{a}_1), \dots, \sigma(\mathbf{a}_\ell))) \\ &\quad = \langle {}_{\sigma_1(v(a_{11},\dots,a_{\ell 1}))}\chi_1^{-1}(v(\sigma_1(a_{11}, \dots, a_{\ell 1}))), \\ &\qquad\qquad \dots, {}_{\sigma_n(v(a_{1n},\dots,a_{\ell n}))}\chi_1^{-1}(v(\sigma_n(a_{1n}, \dots, a_{\ell n})))\rangle \\ &\quad = \langle \mathrm{f}^{\mathcal{E}_1,\sigma_1}[v; \langle a_{11}, \dots, a_{\ell 1}\rangle], \dots, \mathrm{f}^{\mathcal{E}_n,\sigma_n}[v; \langle a_{1n}, \dots, a_{\ell n}\rangle]\rangle. \end{aligned} $$

$\square$

The addition operation on $\mathbf{E_V}(Q,M)$. We observe that $+^M : M^2 \to M$ is an arrow of $\mathbf{Ab}[A,\mathbf{V}]$. Given a pair $\langle \mathcal{E}_1, \mathcal{E}_2\rangle$ of extensions in $\mathbf{V}$ of A by M, $(\mathcal{E}_1 \boxtimes \mathcal{E}_2)\Delta_A$ is an extension of A by ${}_{\Delta_A}\mathrm{Res}\, M^{\boxtimes 2} = M^2$. Thus, we can form the composite $+^M[(\mathcal{E}_1 \boxtimes \mathcal{E}_2)\Delta_A]$, which is a class of extensions in $\mathbf{V}$ of A by M.

Theorem 8.4. *We have*

$$+^M[(\mathcal{E}_1 \boxtimes \mathcal{E}_2)\Delta_A] = [\mathcal{E}_1] + [\mathcal{E}_2].$$

Proof. Let sections σ_1, σ_2 of the extensions $\mathcal{E}_i$ be chosen. The factor set $\mathrm{f}^{\mathcal{E}_1,\sigma_1} + \mathrm{f}^{\mathcal{E}_2,\sigma_2}$ is a factor set of $[\mathcal{E}_1]+[\mathcal{E}_2]$. On the other hand, a factor set for $+^M[(\mathcal{E}_1 \boxtimes \mathcal{E}_2)\Delta_A]$ can be computed from the factor set $\mathrm{f}^{\mathcal{E}_1 \boxtimes \mathcal{E}_2,\sigma}$ computed in theorem 8.3. It is easy to see that these factor sets are equal. □

9. Abelian Extensions as a Bifunctor

We want to be able to treat the $\mathbf{W}$-algebra of abelian extensions of A by M as a bifunctor. On objects, we define the bifunctor

$$\mathbf{E_V} : (\mathbf{Ov}[A,\mathbf{V}])^{\mathrm{op}} \times \mathbf{Ab}[A,\mathbf{V}] \to \mathbf{Ab}$$

by the formula

$$\mathbf{E_V}(Q,M) = \mathbf{E_V}(A \ltimes Q, {}_{\pi_Q}\mathrm{Res}\, M),$$

where we recall that $\pi_Q : A \ltimes Q \to A$ is defined by $\pi_Q : \langle a, q\rangle \mapsto a$.

On arrows, it suffices to define an abelian group homomorphism $\mathbf{E_V}(r, 1_M)$, which we will write as a composition $[\mathcal{E}] \mapsto [\mathcal{E}]r$, and an abelian group homomorphism $\mathbf{E_V}(1_Q, \dot{g})$, which we will write as a composition on the other side, $[\mathcal{E}] \mapsto \dot{g}[\mathcal{E}]$, and to prove the following:

1. $([\mathcal{E}]r)s = [\mathcal{E}](rs)$;
2. $\dot{h}(\dot{g}([\mathcal{E}])) = (\dot{h}\dot{g})[\mathcal{E}]$; and
3. $\dot{g}([\mathcal{E}]g) = (\dot{g}[\mathcal{E}])g$,

when those compositions are defined.

If $\mathcal{E} \in \mathbf{E_V}(Q,M)$, $Q' \in \mathbf{Ov}[A,\mathbf{V}]$, and $r : Q' \to Q$ is a homomorphism, then we define $[\mathcal{E}]r = [\mathcal{E}](A \ltimes r)$. $[\mathcal{E}]r$ is an element of

$$\begin{aligned}\mathbf{E_V}(A \ltimes Q', {}_{A\ltimes r}\mathrm{Res}({}_{\pi_Q}\mathrm{Res}\, M)) &= \mathbf{E_V}(A \ltimes Q', {}_{\pi_{Q'}}\mathrm{Res}\, M)\\ &= \mathbf{E_V}(Q', M).\end{aligned}$$

From corollary 7.4, this mapping is an abelian group homomorphism, and it is easy to see that property (1) holds, from the corresponding fact (corollary 7.3) for composition of extensions with homomorphisms of $\mathbf{V}$.

On the other hand, if M' is another object of $\mathbf{Ab}[A,\mathbf{V}]$ and $\dot{g} : M \to M'$ is a homomorphism, we define

$$\dot{g}[\mathcal{E}] = ({}_{\pi_Q}\mathrm{Res}\, \dot{g})[\mathcal{E}].$$

This is an element of $\mathbf{E_V}(Q, M')$ and it is clear from theorem 7.8 that the mapping is an abelian group homomorphism, and from theorem 7.7 that property (2) holds.

Finally, if $\mathcal{E}$, Q', r, M', and $\dot{g}$ are all given, we must prove that $(\dot{g}[\mathcal{E}])r = \dot{g}([\mathcal{E}]r)$, and this will complete the proof that $\mathbf{E_V}$ is a bifunctor as we have defined it. Using theorem 7.9, we have

$$\begin{aligned}(\dot{g}[\mathcal{E}])r &= (({}_{\pi_Q}\mathrm{Res}\,\dot{g})[\mathcal{E}])(A \ltimes r) \\ &= {}_{A\ltimes r}\mathrm{Res}({}_{\pi_Q}\mathrm{Res}\,\dot{g})([\mathcal{E}](A \ltimes r)) \\ &= {}_{\pi_{Q'}}\mathrm{Res}\,\dot{g}([\mathcal{E}]r) \\ &= \dot{g}([\mathcal{E}]r).\end{aligned}$$

10. Module Extensions

R-modules as a variety of algebras. If R is a ring, then the left R-modules can be treated as algebras having an underlying abelian group structure and one unary operation for each element of R. The reader can easily supply a list of identities defining the variety of left R-modules. This is a congruence-modular variety, as can easily be proved, because of the underlying abelian group structures.

It is important to note for what follows that if $v \in \mathrm{Clo}_n\, R\text{-}\mathbf{Mod}$, then $v(\mathbf{m})$ has the form $\Sigma_i r_i m_i$.

Abelian group overalgebras in the variety of left R-modules. We need to prove some facts about abelian group overalgebras totally in R-**Mod**:

Theorem 10.1. *Let A and B be left R-modules, and $f : A \to B$ an onto homomorphism. Then every abelian group A-overalgebra totally in R-***Mod** *is isomorphic to the restriction, along f, of an abelian group B-overalgebra.*

Proof. Let M be an abelian group A-overalgebra totally in R-**Mod**, and let $\langle M', \eta\rangle$ be a universal arrow to the functor ${}_f\mathrm{Res}$. That is, M' is an induced abelian group B-overalgebra in R-**Mod** of M along f. By [14, theorem 10.6], since f is onto, we can equally well consider M' to be an induced pointed B-overalgebra of M along f. Thus, [14, theorem C.6.5] applies, and the diagram

$$\begin{tikzcd} A \ltimes M \arrow[d, "\mu(A\ltimes\eta_M)"'] & A \arrow[l, "\iota_M"'] \arrow[d, "f"] \\ B \ltimes M' & B \arrow[l, "\iota_{M'}"'] \end{tikzcd}$$

is a pushout diagram, where μ is the *mashing homomorphism* defined by $\mu : \langle a, x\rangle \mapsto \langle f(a), x\rangle$. However, $\pi_M \iota_M = 1_A$, whence $A \ltimes M \cong A \oplus {}_0M$. Similarly, $B \ltimes M' \cong B \oplus {}_0M'$. It follows easily that ${}_0M \cong {}_0M'$, and that $M \cong {}_f\mathrm{Res}\, M'$. □

Corollary 10.2. *If M is an abelian group A-overalgebra totally in R-***Mod***, then M is isomorphic to a restriction, along the unique homomorphism $\pi_0 : A \to 0$, of an abelian group 0-overalgebra.*

Corollary 10.3. *If M, M' are abelian group A-overalgebras totally in R-$\mathbf{Mod}$, and $\chi : M \to M'$ is a homomorphism, then χ is determined by ${}_0\chi$.*

Proof. By [14, theorem C.6.4] restriction along π is a full functor from $\mathbf{Pnt}[0, R\text{-}\mathbf{Mod}]$ to $\mathbf{Pnt}[A, R\text{-}\mathbf{Mod}]$. Since M and M' are isomorphic to restrictions along π, $\chi = \phi^{-1} \circ ({}_\pi\mathrm{Res}\,\hat{\chi}) \circ \phi'$, where $\phi : M \cong {}_\pi\mathrm{Res}\,\hat{M}$, $\phi' : M' \cong {}_\pi\mathrm{Res}\,\hat{M}'$, and $\hat{\chi} : \hat{M} \to \hat{M}'$. It follows that χ is determined by any of its components ${}_a\chi$. ☐

Abelian extensions and module extensions.

Theorem 10.4. *Let $Q \in \mathbf{Ov}[0, R\text{-}\mathbf{Mod}]$, and let $M \in \mathbf{Ab}[0, R\text{-}\mathbf{Mod}]$. Then $\mathbf{E_V}(Q, M) \cong \mathrm{Ext}(0 \ltimes Q, {}_0M)$.*

Proof. Given an extension $\langle \chi, E, \pi \rangle$ of $0 \ltimes Q$ by ${}_{\pi_Q}\mathrm{Res}\,M$, let $\iota = {}_0\chi$, and view ι as a homomorphism from ${}_0M$ to E. Then we have a module extension

$$0 \to {}_0M \xrightarrow{\iota} E \xrightarrow{\pi} 0 \ltimes Q \to 0$$

of $0 \ltimes Q$ by ${}_0M$.

On the other hand, given ι, E, and π, we define ${}_e\chi(m) = e + \iota(m)$. This gives an E-function which is clearly one-one and onto, and is an E-overalgebra homomorphism from ${}_{\pi_Q\pi}\mathrm{Res}\,M$ to κ^*, where $\kappa = \ker \pi$, because

$$\begin{aligned}
v_{\mathbf{e}}^{\kappa^*}({}_{\mathbf{e}}\chi(\mathbf{m})) &= v_{\mathbf{e}}^{\kappa^*}(e_1 + \iota(m_1), \ldots, e_n + \iota(m_n)) \\
&= v(e_1 + \iota(m_1), \ldots, e_n + \iota(m_n)) \\
&= v(\mathbf{e}) + v(\iota(\mathbf{m})) \\
&= v(\mathbf{e}) + \iota(v^M(\mathbf{m})) \\
&= v(\mathbf{e}) + \iota(v^M_{\langle 0,\ldots,0\rangle}(\mathbf{m})) \\
&= {}_{v(\mathbf{e})}\chi(v^M_{\langle 0,\ldots,0\rangle}(\mathbf{m})) \\
&= {}_{v(\mathbf{e})}\chi(v_{\mathbf{e}}^{\pi_Q\pi\mathrm{Res}\,M}(\mathbf{m}));
\end{aligned}$$

thus, χ is an isomorphism, and $\langle \chi, E, \pi \rangle$ is an extension in R-$\mathbf{Mod}$ of $0 \ltimes Q$ by ${}_{\pi_Q}\mathrm{Res}\,M$, i.e., an element of $\mathbf{E_V}(Q, M)$.

The first construction, of ι from χ, is one-one by corollary 10.3, but it is also onto, because if we start with ι and construct χ, then we get ι back as $\iota = {}_0\chi$. Thus, the two mappings are inverses to each other.

If two extensions $\mathcal{E} = \langle \chi, E, \pi \rangle$ and $\tilde{\mathcal{E}} = \langle \tilde{\chi}, \tilde{E}, \tilde{\pi} \rangle$ in R-$\mathbf{Mod}$ of $0 \ltimes Q$ by M are given, and $\gamma : \mathcal{E} \sim \tilde{\mathcal{E}}$, then from $\gamma^*\chi = {}_\gamma\mathrm{Res}\,\tilde{\chi}$ we obtain that $\gamma_0\chi = {}_0\gamma^*{}_0\chi = {}_0\tilde{\chi}$, whence we have a commutative diagram

$$\begin{array}{ccccc}
{}_0M & \xrightarrow{{}_0\chi} & E & \xrightarrow[\pi]{} & 0 \ltimes Q \\
\| & & \downarrow{\scriptstyle\gamma} & & \| \\
{}_0M & \xrightarrow[{}_0\tilde{\chi}]{} & \tilde{E} & \xrightarrow[\tilde{\pi}]{} & 0 \ltimes Q
\end{array}$$

showing that the constructed module extensions remain equivalent, by the customary definition. On the other hand, given a commutative diagram

$$\begin{tikzcd}
{}_0M \arrow[r, "\iota"] \arrow[d, equal] & E \arrow[r, "\pi"'] \arrow[d, "\gamma"'] & 0 \ltimes Q \arrow[d, equal] \\
{}_0M \arrow[r, "\tilde{\iota}"'] & \tilde{E} \arrow[r, "\tilde{\pi}"'] & 0 \ltimes Q
\end{tikzcd}$$

then let χ and $\tilde{\chi}$ be constructed from ι and $\tilde{\iota}$: we have $\tilde{\pi}\gamma = \pi$ and for all $e \in E$, and $m \in {}_0M$,

$$\begin{aligned}
{}_e\gamma^* {}_e\chi(m) &= {}_e\gamma^*(e + \iota(m)) \\
&= \gamma(e) + \gamma\iota(m) \\
&= \gamma(e) + \tilde{\iota}(m) \\
&= {}_{\gamma(e)}\tilde{\chi}(m) \\
&= {}_e({}_\gamma\mathrm{Res}\,\tilde{\chi})(m),
\end{aligned}$$

whence $\gamma^*\chi = {}_\gamma\mathrm{Res}\,\tilde{\chi}$. Thus, the constructed elements of $\mathbf{E}_\mathbf{V}(Q, M)$ are equivalent by our definition.

We omit the verifications that this isomorphism between the bifunctors $\mathbf{E}_\mathbf{V}(Q, M)$ and $\mathrm{Ext}(0 \ltimes Q, {}_0M)$ of M and Q is natural in both arguments, and that it preserves the abelian group operations. $\square$

11. Clone Cohomology

The partial cochain complex which we introduced in §6, leading to the cohomology group we have identified as the set of extensions $\mathbf{E}_\mathbf{V}(A, M)$, can be extended to a full positive cochain complex (i.e., with objects $C^i \in \mathbf{Ab}[A, \mathbf{V}]$ for all $i \geq 0$) and we will do so in this section. As we do, we want to make some small changes in our development. We want to develop the theory in such a way that, as we did in defining $\mathbf{E}_\mathbf{V}(Q, M)$ in §9, the resulting cohomology objects are bifunctors contravariant in $\mathbf{Ov}[A, \mathbf{V}]$ and covariant in $\mathbf{Ab}[A, \mathbf{V}]$. In addition, we want to express the cochain complex as a result of applying a hom functor to a chain complex we will define. Finally, we want to make explicit the simplicial aspect of the definition. We will call the resulting cohomology theory *clone cohomology*, because of the role played in the definition by the clone of the variety $\mathbf{V}$.

The A-sets $X^i_\mathbf{V}(Q)$. As a first step, we will define functors $X^i_\mathbf{V} : \mathbf{Ov}[A, \mathbf{V}] \to A\text{-}\mathbf{Set}$. The A-sets $X^i_\mathbf{V}(A)$, for $i = 0$, 1, and 2, can be seen as the special cases of $X^i_\mathbf{V}(Q)$ where Q is the A-set $[\![A, 1_A]\!]$.

If Q is an object of $\mathbf{Ov}[A, \mathbf{V}]$, we will define $X^0_\mathbf{V}(Q)$ to be the underlying A-set of $[\![A \ltimes Q, \pi_Q]\!]$, with the element $\langle a, q\rangle \in {}_aX^0_\mathbf{V}(Q)$ being written as $[q]_a$.

We define $X^1_\mathbf{V}(Q)$ to be the A-set given by letting ${}_aX^1_\mathbf{V}(Q)$ be the set of triples, written $[v; \mathbf{q}]_\mathbf{a}$, where $v \in \mathrm{Clo}_n\,\mathbf{V}$ for some n, $\mathbf{a} \in A^n$, and $\mathbf{q} \in {}_\mathbf{a}Q^{\boxtimes n}$, and such that $v(\mathbf{a}) = a$.

If $i > 1$, we will define ${}_aX^i_{\mathbf{V}}(Q)$ to be the set of $(i+2)$-tuples, written

$$[v_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}},$$

where v_0 is an element of $\mathrm{Clo}_{n_0} \mathbf{V}$, $\mathbf{v}_j$ for $0 < j < i$ is an n_{j-1}-tuple of elements of $\mathrm{Clo}_{n_j}(\mathbf{V})$, $\mathbf{a} \in A^{n_{i-1}}$, and $\mathbf{q} \in {}_{\mathbf{a}}Q^{\boxtimes n_i}$, all for some natural numbers $n_0, \dots, n_{i-1}$, and such that $v_0\mathbf{v}_1 \dots \mathbf{v}_{i-1}(\mathbf{a}) = a$.

Note that v_0, $\mathbf{v}_1$, $\dots$, $\mathbf{v}_{i-1}$ can be considered as composable arrows of the theory constructed from $\mathrm{Clo}\, \mathbf{V}$.

∂. First, we will define ∂ directly, making $A\text{-}\mathbf{Set}(X^\bullet_{\mathbf{V}}(Q), M)$ into a chain complex. Given an A-function $\mathrm{f} : X^i_{\mathbf{V}}(Q) \to M$, we define $\partial \mathrm{f}$ by

$$(\partial \mathrm{f})[v; \mathbf{q}]_{\mathbf{a}} = v^M_{\mathbf{a}} \mathrm{f}[\mathbf{q}]_{\mathbf{a}} - \mathrm{f}[v^Q_{\mathbf{a}}(\mathbf{q})]_{v(\mathbf{a})}$$

if $i = 0$, where $[\mathbf{q}]_{\mathbf{a}}$ stands for $\langle [q_1]_{a_1}, \dots, [q_n]_{a_n} \rangle$, and otherwise, by

$$\begin{aligned}(\partial \mathrm{f})[v_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} = {} & (v_0)^M_{\mathbf{v}_1 \dots \mathbf{v}_{i-1}(\mathbf{a})} \mathrm{f}[\mathbf{v}_1, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} \\ & - \mathrm{f}[v_0\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} \\ & + \mathrm{f}[v_0, \mathbf{v}_1\mathbf{v}_2, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} \\ & \dots \\ & + (-1)^j \mathrm{f}[v_0, \mathbf{v}_1, \dots, \mathbf{v}_{j-1}\mathbf{v}_j, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} \\ & \dots \\ & + (-1)^i \mathrm{f}[v_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-2}; (\mathbf{v}_{i-1})^Q_{\mathbf{a}}(\mathbf{q})]_{\mathbf{v}_{i-1}(\mathbf{a})},\end{aligned}$$

where $\mathrm{f}[\mathbf{v}_1, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}}$ stands for $\langle \mathrm{f}[v_{11}, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}}, \dots, \mathrm{f}[v_{1n_0}, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} \rangle$.

A straightforward computation shows that $\partial\partial = 0$, making $A\text{-}\mathbf{Set}(X^\bullet_{\mathbf{V}}(Q), M)$ into a cochain complex, which we denote by $C^\bullet_{\mathbf{V}}(Q, M)$..

Another way to arrive at the same cochain complex is to apply the free functor from A-$\mathbf{Set}$ to $\mathbf{Ab}[A, \mathbf{V}]$ to the A-sets $X^i_{\mathbf{V}}(Q)$, yielding objects $C_i(Q, \mathbf{V}) \in \mathbf{Ab}[A, \mathbf{V}]$. We then consider the simplicial complex of objects of $\mathbf{Ab}[A, \mathbf{V}]$ given by the $C_i(Q, \mathbf{V})$ and face maps ∂^i_j, for $i \geq 0$ and $0 \leq j \leq i$, defined on generators by

$$\partial^0_0[q]_a = 0,$$

by

$$\partial^1_0[v; \mathbf{q}]_{\mathbf{a}} = v^M_{\mathbf{a}}[\mathbf{q}]_{\mathbf{a}} \quad \text{and} \quad \partial^1_1[v; \mathbf{q}]_{\mathbf{a}} = [v^Q_{\mathbf{a}}(\mathbf{q})]_{v(\mathbf{a})},$$

and by

$$\partial^i_j[v_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}} = \begin{cases} (v_0)^M_{\mathbf{v}_1 \dots \mathbf{v}_{i-1}(\mathbf{a})}[\mathbf{v}_1, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}}, & \text{for } j = 0, \\ [v_0, \mathbf{v}_1, \dots, \mathbf{v}_{j-1}\mathbf{v}_j, \dots, \mathbf{v}_{i-1}; \mathbf{q}]_{\mathbf{a}}, & \text{for } 0 < j < i, \text{ and} \\ [v_0, \mathbf{v}_1, \dots, \mathbf{v}_{i-2}; (\mathbf{v}_{i-1})^Q_{\mathbf{a}}(\mathbf{q})]_{\mathbf{v}_{i-1}(\mathbf{a})}, & \text{for } j = i, \end{cases}$$

for $i \geq 2$. (Edge maps can also be defined, using projection elements of $\operatorname{Clo} \mathbf{V}$, but we have no need to do so.) We then form ∂ as the alternating sum of the face maps ∂^i_j over j, and obtain a chain complex $C_\bullet(Q, \mathbf{V})$ such that $\mathbf{Ab}[A, \mathbf{V}](C_\bullet(Q, \mathbf{V}), M) = C^\bullet_{\mathbf{V}}(Q, M)$, the same cochain complex we described previously.

Clone cohomology. We define the *clone cohomology objects for Q, with coefficients in M*, to be the cohomology groups of the cochain complex $C^\bullet_{\mathbf{V}}(Q, M)$, and denote them by $H^i_{\mathbf{V}}(Q, M)$. Clearly, they are bifunctors $H^i : \mathbf{Ov}[A, \mathbf{V}]^{\mathrm{op}} \times \mathbf{Ab}[A, \mathbf{V}] \to \mathbf{Ab}$.

Note that the clone cohomology objects are defined whether or not the variety $\mathbf{V}$ is congruence-modular.

Factor sets in terms of Q. We previously defined factor sets of extensions of A by M, and then defined extensions of an A-overalgebra Q by M. We have not defined factor sets in terms of Q yet, and will not do so in detail. The key fact which allows us to relate our previous work with factor sets, and the objects $X^\bullet_{\mathbf{V}}(Q)$ and $C^\bullet_{\mathbf{V}}(Q, M)$, is as follows:

Theorem 11.1. *For $i = 0$, 1, and 2, the three functors to $\mathbf{Ab}[A, \mathbf{V}]$,*

$$A\text{-}\mathbf{Set}(X^i_{\mathbf{V}}(A \ltimes Q), {}_{\pi_Q}\operatorname{Res} M),$$

$$A\text{-}\mathbf{Set}(X^i_{\mathbf{V}}(Q), M), \text{ and}$$

$$C^i_{\mathbf{V}}(Q, M)$$

are naturally isomorphic as bifunctors in Q and M.

Proof. The first and second functors are naturally isomorphic, because of an adjunction between the functor of restriction of A-sets along π_Q and the functor from $(A \ltimes Q)$-sets to A-sets given by sending an $(A \ltimes Q)$-set $[\![B, \pi]\!]$ to the A-set $[\![B, \pi_Q \circ \pi]\!]$.

The second and third functors are naturally isomorphic, because of the adjunction between the free functor from A-$\mathbf{Set}$ to $\mathbf{Ab}[A, \mathbf{V}]$, and the corresponding forgetful functor. ☐

12. Relative Clone Cohomology V

The clone cohomology functors $H^i_{\mathbf{V}}(Q, M)$ are specific to a given variety of algebras $\mathbf{V}$, such that Q and M are totally in $\mathbf{V}$. However, the construction of the chain complex used in computing them can use a smaller variety $\mathbf{V}'$, if M is totally in $\mathbf{V}'$. That is, in that case, we can form the free object in $\mathbf{Ab}[A, \mathbf{V}']$ on A-set of generators $X^i_{\mathbf{V}}(Q)$, which we denote by $C_{i,\mathbf{V}'}(Q, \mathbf{V})$, rather than the free object in $\mathbf{Ab}[A, \mathbf{V}]$ which we denoted by $C_i(Q, \mathbf{V})$. Because M is totally in $\mathbf{V}'$, the hom functor $\mathbf{Ab}[A, \mathbf{V}](-, M)$ takes these objects to the same abelian group. The definition of ∂ also makes sense. As a result,

Theorem 12.1. *If M is totally in $\mathbf{V}'$, then the objects $C^i_{\mathbf{V}}(Q, M)$ and $H^i_{\mathbf{V}}(Q, M)$ are also totally in $\mathbf{V}'$.*

If Q is also totally in $\mathbf{V}'$, then the definition of ∂ for the chain complex $C_\bullet(Q,\mathbf{V}')$ makes sense, and there is an obvious homomorphism π_i of $C_i(Q,\mathbf{V})$ onto $C_i(Q,\mathbf{V}')$ for each i. As a result, we have a short exact sequence of complexes in $\mathbf{Ab}[A,\mathbf{V}]$,

$$\begin{tikzcd}
K_0(Q,\mathbf{V}',\mathbf{V}) \arrow[d] & K_1(Q,\mathbf{V}',\mathbf{V}) \arrow[l] \arrow[d] & K_2(Q,\mathbf{V}',\mathbf{V}) \arrow[l] \arrow[d] & \dots \arrow[l] \\
C_{0,\mathbf{V}'}(Q,\mathbf{V}) \arrow[d, "\pi_0"] & C_{1,\mathbf{V}'}(Q,\mathbf{V}) \arrow[l] \arrow[d, "\pi_1"] & C_{2,\mathbf{V}'}(Q,\mathbf{V}) \arrow[l] \arrow[d, "\pi_2"] & \dots \arrow[l] \\
C_0(Q,\mathbf{V}') & C_1(Q,\mathbf{V}') \arrow[l] & C_2(Q,\mathbf{V}') \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

where the $K_i(Q,\mathbf{V}',\mathbf{V})$ are the kernels of the onto homomorphisms π_i. Note that the vertical short exact sequences split. For, each generator of $C_i(Q,\mathbf{V}')$ can be mapped to a preimage in $C_{i,\mathbf{V}'}(Q,\mathbf{V})$, leading to a splitting of π_i.

Applying the hom functor $\mathbf{Ab}[A,\mathbf{V}](-,M)$ and denoting $\mathbf{Ab}[A,\mathbf{V}](K_i(Q,\mathbf{V}',\mathbf{V}),M)$ by $C^i_{\mathbf{V}',\mathbf{V}}(Q,M)$, we obtain a short exact sequence of complexes of objects of $\mathbf{Ab}[A,\mathbf{V}']$:

$$\begin{tikzcd}
C^0_{\mathbf{V}',\mathbf{V}}(Q,M) \arrow[r] & C^1_{\mathbf{V}',\mathbf{V}}(Q,M) \arrow[r] & C^2_{\mathbf{V}',\mathbf{V}}(Q,M) \arrow[r] & \dots \\
C^0_{\mathbf{V}}(Q,M) \arrow[r] \arrow[u] & C^1_{\mathbf{V}}(Q,M) \arrow[r] \arrow[u] & C^2_{\mathbf{V}}(Q,M) \arrow[r] \arrow[u] & \dots \\
C^0_{\mathbf{V}'}(Q,M) \arrow[r] \arrow[u] & C^1_{\mathbf{V}'}(Q,M) \arrow[r] \arrow[u] & C^2_{\mathbf{V}'}(Q,M) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

where, again, the vertical sequences are exact.

We define the *relative clone cohomology of Q, with coefficients in M, with respect to the inclusion* $\mathbf{V}' \subseteq \mathbf{V}$, to be the cohomology objects of the complex $C^\bullet_{\mathbf{V}',\mathbf{V}}(Q,M)$, and denote these objects by $H^i_{\mathbf{V}',\mathbf{V}}(Q,M)$. It is clear that $C^0_{\mathbf{V}}(Q,M)$ and $C^0_{\mathbf{V}'}(Q,M)$ are isomorphic; thus, $C^0_{\mathbf{V}',\mathbf{V}}(Q,M) = H^0_{\mathbf{V}',\mathbf{V}}(Q,M) = 0$. We then see that there is a long exact sequence of cohomology groups

$$0 \to H^1_{\mathbf{V}'}(Q,M) \to H^1_{\mathbf{V}}(Q,M) \to H^1_{\mathbf{V}',\mathbf{V}}(Q,M) \to H^2_{\mathbf{V}'}(Q,M) \to \dots$$

relating the three sets of cohomology objects $H^\bullet_{\mathbf{V}'}(Q,M)$, $H^\bullet_{\mathbf{V}}(Q,M)$, and $H^\bullet_{\mathbf{V}',\mathbf{V}}(Q,M)$.

If we have three varieties $\mathbf{V}'' \subseteq \mathbf{V}' \subseteq \mathbf{V}$ such that Q and M are totally in $\mathbf{V}''$, then similar methods, and standard methods from homological algebra, yield a long exact sequence

$$0 \to H^1_{\mathbf{V}'',\mathbf{V}'}(Q,M) \to H^1_{\mathbf{V}'',\mathbf{V}}(Q,M) \to H^1_{\mathbf{V}',\mathbf{V}}(Q,M) \to H^2_{\mathbf{V}'',\mathbf{V}'}(Q,M) \to \dots$$

relating the relative cohomology objects.

Discussion

The study of abelian extensions, and the recognition that they form a cohomology group, has a considerable history, as does the investigation of cohomology theories in general. We

have focussed our attention on the abelian group of extensions, and have defined a cohomology theory that has this algebra as cohomology group in dimension one. Many questions remain to be answered about this new cohomology theory, and about its relationship to previously-studied theories of cohomology of algebras. We will raise some of those questions in this section.

Categorical algebraists have invented a cohomology theory called comonadic cohomology. The theory can be applied to any comonad, but usually, the comonad being used when this theory is mentioned is a comonad derived from the free and forgetful functors for the variety in question. See [1], [3], and [2]. As in clone cohomology, the theory gives the cohomology group of an A-overalgebra (actually, an algebra over A, as the theory is usually developed) totally in a variety $\mathbf{V}$ to which A belongs, with coefficients in an A-module M totally in $\mathbf{V}$ (or, as usually expressed, in a Beck module over A). As in clone cohomology, the resolution that gives rise to the cohomology objects comes from taking alternating sums of face maps of a simplicial complex.

We are led to ask, what is the relationship of the clone cohomology objects to the comonadic cohomology objects? In [3], the comonadic cohomology groups in dimension 0 and 1 were studied and interpreted. This was done in the generality of $\mathbf{V}$ an arbitrary (i.e., not necessarily congruence-modular) variety of algebras of some type. The interpretation of the group in dimension 1 is different from our interpretation of the clone cohomology group in dimension 1, but, we have proved directly that, for $\mathbf{V}$ a congruence-modular variety of algebras, the groups are naturally isomorphic. We have not discussed dimension 0 in this paper, but the groups in dimension 0 are also isomorphic. (We have not included the proofs of these results in this paper.)

This raises the question, are the groups isomorphic in all dimensions, when $\mathbf{V}$ is congruence-modular? More generally, are they isomorphic when $\mathbf{V}$ is not congruence-modular? It is not hard to show that they are isomorphic in dimension 0, but the answer is not known for higher dimensions, or even for dimensions higher than 1 in the congruence-modular case.

For clone cohomology, and for comonadic cohomology for that matter, there is the question, when are the resolutions used in the derivation of the cohomology groups exact? Note that in both cases, the resolutions are free. Thus, when the resolutions are exact, the cohomology functors will be derived functors in the standard sense. Otherwise, for comonadic cohomology, there is at least a uniqueness theorem ([2]) which characterizes the cohomology functors. For clone cohomology, there is not yet such a theorem.

For some well-known cases, such as the variety of groups, the comonadic and clone cohomology groups coincide with the standard cohomology groups which can be defined as derived functors in the usual sense. Thus, we can say that we have characterized these groups using two different universal properties. What is the significance of the fact that the two different universal properties arrive at the same answer?

Finally, we should mention that there is a formal, at least, resemblence between the derivation of the clone cohomology objects, and the construction of the cohomology groups of a category. What is the significance of this resemblence?

To our knowledge, none of the questions have yet been answered.

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48 Shattuck Square, #86, Berkeley, California 94704
E-mail address: `rowan@crl.com`